

LIE ALGEBRA REPR THEORY III: SPINOR IRREPS OF SO(N) AND CLIFFORD ALGEBRAS

BRIEF REVIEW: ELEMENTS OF LIE ALGEBRA THEORY

CHEVALLEY BASIS:

$$\begin{aligned}
 e_i &\equiv e_{\alpha_i} & [h_i, e_j] &= A_{ji} e_j \\
 f_i &\equiv e_{-\alpha_i} & [h_i, f_j] &= -A_{ji} f_j \\
 h_i &\equiv h_{\alpha_i^\vee} & [e_i, f_j] &= \delta_{ij} \frac{|\alpha_i|^2}{2} (e_{\alpha_i}, e_{-\alpha_i}) h_i
 \end{aligned}$$

IRREP: $\Lambda_{HWS} = \sum_{j=1}^n K_j^{(\Lambda)} \alpha_j = \sum_{j=1}^n \Lambda_j \omega_j$
BANK n.l.a.

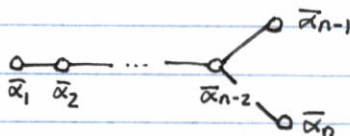
WEIGHT VECTOR ϕ_Λ : $\hat{H}_i \phi_\Lambda = \langle \Lambda, \alpha_i^\vee \rangle \phi_\Lambda = \Lambda_i \phi_\Lambda$
DYKIN COEFF. ARE CSA EVALS IN THE CHEVALLEY BASIS

CASIMIR $\hat{C} = \sum_{i,j=1}^n \langle \alpha_i, \alpha_j \rangle_2^{-1} \hat{H}_{\alpha_i}^{(2)} \hat{H}_{\alpha_j}^{(2)} + \sum_{\alpha \neq 0} (e_\alpha, e_{-\alpha})_2 \hat{E}_\alpha^{(2)} \hat{E}_{-\alpha}^{(2)} = \sum_a \hat{T}^a \hat{T}^a$

$TR_\Lambda[\hat{C}] = N_\Lambda \langle \Lambda, \Lambda + 2\rho \rangle_2 = N_{Adj} \lambda_\Lambda \Rightarrow \lambda_\Lambda = \frac{N_\Lambda}{N_{Adj}} \langle \Lambda, \Lambda + 2\rho \rangle_2$
INDEX (TR. NORM): $TR_\Lambda[\hat{T}^a \hat{T}^b] = \lambda_\Lambda \delta^{ab}$

ADJOINT: $\lambda_\theta = \langle \theta, \theta + 2\rho \rangle_2 = 2g$

GEO 4, p. 71-75 EX: SO(2n)



FUNDAMENTAL IRREP:

$h_{\alpha_i^\vee} \rightarrow \hat{E}_{ii}^1 - \hat{E}_{i+1, i+1}^1, \quad 1 \leq i \leq n-1$, $\hat{E}_{jk}^1 \equiv \hat{E}_{jk} - \hat{E}_{k+n, j+n} \rightarrow$

$h_{\alpha_n^\vee} \rightarrow \hat{E}_{n-1, n-1}^1 + \hat{E}_{n, n}^1$

$\hat{E}_{jk}^2 \equiv \hat{E}_{j, n+k} - \hat{E}_{k, n+j} \rightarrow$

ex: so(8)

$$\begin{aligned}
 \hat{H}_{\alpha_1^\vee} &= \text{diag}(1, -1, 0, 0, -1, 1, 0, 0) \\
 \hat{H}_{\alpha_2^\vee} &= \text{diag}(0, 1, -1, 0, 0, -1, 1, 0) \\
 \hat{H}_{\alpha_3^\vee} &= \text{diag}(0, 0, 1, -1, 0, 0, -1, 1) \\
 \hat{H}_{\alpha_4^\vee} &= \text{diag}(0, 0, 0, 1, 1, 0, 0, -1)
 \end{aligned}$$

$$\begin{aligned}
 \hat{E}_{-\alpha_1} &= \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \\
 \hat{E}_{-\alpha_2} &= \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \\
 \hat{E}_{-\alpha_3} &= \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \\
 \hat{E}_{-\alpha_4} &= \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}
 \end{aligned}$$

SIMPLE ROOT LOWERING OPERATORS

⇒ GEN CSA : ELEM

$$\hat{H} = \begin{bmatrix} \Lambda^{(1)} & & & & \\ & \Lambda^{(2)} & & & \\ & & \Lambda^{(3)} & & \\ & & & \Lambda^{(4)} & \\ & & & & -\Lambda^{(1)} \\ & & & & & -\Lambda^{(2)} \\ & & & & & & -\Lambda^{(3)} \\ & & & & & & & -\Lambda^{(4)} \end{bmatrix}$$

GEO 4, p. 90E

SO(8)

1 0 0 0 ϕ_1

↓ $\bar{\alpha}_1$

-1 1 0 0 ϕ_2

↓ $\bar{\alpha}_2$

0 -1 1 1 ϕ_3

$\bar{\alpha}_3$ ↙ ↘ $\bar{\alpha}_4$

ϕ_4 0 0 -1 1 0 0 1 -1 $\phi_5 = \bar{\phi}_8$

↘ $\bar{\alpha}_4$ ↙ $\bar{\alpha}_3$

0 1 -1 -1 $\phi_6 = \bar{\phi}_7$

↓ $\bar{\alpha}_2$

1 -1 0 0 $\phi_7 = \bar{\phi}_6$

↓ $\bar{\alpha}_1$

-1 0 0 0 $\phi_8 = \bar{\phi}_5$

MORE CONV. ORDERING

$$\phi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

; $\Lambda^{(1)} \equiv \{1, 0, 0, 0\}_D$ ← DYNKIN
 ← A SIMULTANEOUS EIGENVECTOR OF $\{\hat{H}_{\bar{\alpha}_1}, \hat{H}_{\bar{\alpha}_2}, \hat{H}_{\bar{\alpha}_3}, \hat{H}_{\bar{\alpha}_4}\}$

$$\phi_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

; $\Lambda^{(2)} = \{-1, 1, 0, 0\}_D$

$$\phi_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

; $\Lambda^{(3)} = \{0, -1, 1, 1\}_D$

$$\phi_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \Lambda^{(4)} = \{0, 0, -1, 1\}_D$$

$$\phi_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \Lambda^{(5)} = \{0, 0, 1, -1\}_D = -\Lambda^{(4)}$$

$$\phi_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}; \Lambda^{(6)} = \{0, 1, -1, -1\} = -\Lambda^{(3)}$$

$$\phi_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \Lambda^{(7)} = \{1, -1, 0, 0\} = -\Lambda^{(2)}$$

$$\phi_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \Lambda^{(8)} = \{-1, 0, 0, 0\} = -\Lambda^{(1)}$$

THERE ARE THREE NATURAL BASES FOR THE WEIGHT SPACE H_0^*

① SIMPLE ROOTS $\Lambda = \sum_{j=1}^n k_j^{(n)} \bar{\alpha}_j$; $\langle \bar{\alpha}_i, \bar{\alpha}_j \rangle = A_{ij}$
 (Typically non-integer unless in the same conv. class as the adjoint)

② FUND. WEIGHTS $\Lambda = \sum_{j=1}^n \Lambda_j \omega_j$; $\langle \omega_i, \omega_j \rangle = F_{ij}$
 (Dynkin)

NEW * ③ WEIGHTS OF THE FUND. IRREP. $\Lambda = \sum_{j=1}^n \Gamma_j \Lambda^{(j)}$; $\langle \Lambda^{(j)}, \Lambda^{(k)} \rangle = M^{jk}$
 (Which weight ⇒ this is NOT a Dynkin label)

LET'S COMPUTE $\langle \Lambda^{(j)}, \Lambda^{(k)} \rangle$ FOR $SO(8)$:

so(2n)

$$\begin{aligned} \Lambda^{(1)} &= \{1000\}_D \\ \Lambda^{(2)} &= \{-1100\}_D \\ \Lambda^{(3)} &= \{0-110\}_D \\ \Lambda^{(4)} &= \{0-111\}_D \end{aligned}$$

$$\langle \Lambda^{(j)}, \Lambda^{(k)} \rangle = \sum_{m,n=1}^4 \Lambda_m^{(j)} \Lambda_n^{(k)} F_{mn}$$

CFT BOOK, CH 13:

$$F = \frac{1}{2} \begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

RECALL: QUAD FORM MTX ENTIRELY
DET BY DYKIN DIAGRAM
 $(F^{-1})_{ij} = A_{ij} \frac{2}{|\alpha_i|^2}$
GEO 42, p.1

MATHEMATICA: $\langle \Lambda^{(j)}, \Lambda^{(k)} \rangle = \delta^{ij}$ ORTHONORMAL!

CLAIM: GENERALIZES TO $SO(2n) \Rightarrow \{\Lambda^{(j)}\}$ ARE ORTHONORMAL BASIS IN ROOT/WEIGHT SPACE H_0^*

GEO 4, p. 72:

WE CAN EXPRESS THE ROOTS IN TERMS OF $\{\Lambda^{(j)}\}$

↑ EVALS OF GEN (SA ELEM IN FUND. REP.!) $H = \begin{bmatrix} \Lambda^{(1)} & & & \\ & \Lambda^{(2)} & & \\ & & \Lambda^{(3)} & \\ & & & \Lambda^{(4)} \end{bmatrix}$

I. POSITIVE ROOTS

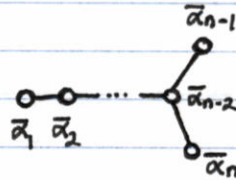
$$\begin{aligned} \textcircled{1} \alpha_{jk}^1 &= \Lambda^{(j)} - \Lambda^{(k)} \iff [_{jk}^1] , j < k \\ \textcircled{2} \alpha_{jk}^2 &= \Lambda^{(j)} + \Lambda^{(k)} \iff [_{jk}^2] , j < k \end{aligned}$$

$\frac{n(n-1)}{2}$ OF THESE
 $\frac{n(n-1)}{2}$ OF THESE

$$\Rightarrow 2[n(n-1)] + n = 2n^2 - n = \frac{1}{2} 2n(2n-1) \checkmark$$

II. SIMPLE ROOTS

$$\begin{aligned} \textcircled{1} \bar{\alpha}_1 &= \Lambda^{(1)} - \Lambda^{(2)} \\ \bar{\alpha}_2 &= \Lambda^{(2)} - \Lambda^{(3)} \\ &\vdots \\ \bar{\alpha}_{n-1} &= \Lambda^{(n-1)} - \Lambda^{(n)} \end{aligned}$$



\Rightarrow CLEARLY, $|\bar{\alpha}_i|^2 = 2 \forall i \in \{1, \dots, n\}$ \checkmark $SO(2n)$ IS SIMPLY-LACED

$\bar{\alpha}_i = \bar{\alpha}_i^\vee \forall i$

$$\textcircled{2} \bar{\alpha}_n = \Lambda^{(n-1)} + \Lambda^{(n)} = \alpha_{n-1, n}^2$$

III. FUND. WEIGHTS

$$\textcircled{1} \omega_i = \sum_{j=1}^i \Lambda^{(j)} , 1 \leq i \leq n-2 \quad \text{C.F. GEO 42, p. 16}$$

$$\textcircled{2} \omega_{n-1} = \frac{1}{2} \sum_{j=1}^{n-1} \Lambda^{(j)} - \frac{1}{2} \Lambda^{(n)} ; \quad \textcircled{3} \omega_n = \frac{1}{2} \sum_{j=1}^n \Lambda^{(j)}$$