PHYS 600 Modern Mathematical Physics I:
Mainly Lie algebra representation theory (Fall 2016)

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Lectures: 9:25-10:40 AM T. Th., HBH 423
Office hours: 10 AM–noon F, 303 Brockman Hall

Introduction: This is a class about modern (post 19th century) mathematical methods that are useful in physics. We will focus on Lie algebras and their representations. These are generators of continuous symmetry transformations that generalize the familiar group of rotations in 3D [SO(3)] to arbitrarily high dimensions. Whenever a symmetry is present, its exploitation simplifies a physics problem. It is routine to encounter higher-dimensional Lie algebras in physical theories, from the su(3) of quark colors to *emergent* so(5) or su(n) symmetries in condensed matter physics. But the connection between continuous symmetries and quantum mechanics runs much deeper than this. Indeed, an elementary particle in quantum field theory is precisely defined as an irreducible representation of the Lorentz group. Many frontier topics in both high energy and condensed matter rely on principles of holography, where topological or gravitational physics in a higher dimensional "bulk" is dual to a special, highly symmetrical (conformal field) theory living on the "boundary." Extensions of Lie algebra representations to affine algebras and quantum groups bring together 2D conformal field theory, 3D topological field theory, anyons, the fractional quantum Hall effect, holography, and exactly solvable quantum many-particle systems. Thus the motivations for learning something about Lie algebras are legion.

That said, this is a *math methods* class, and there will be little if any actual physics discussed. This is partly to keep the material accessible, because while the math itself is universal, specific applications to physics problems can require deep knowledge and intuition within a given field. The main goal of this course is instead to help remove language and conceptual barriers, and to teach elementary manipulations with the structures that one encounters in representations. Most of the course will focus on ordinary (finite dimensional) Lie algebras, although we may touch infinite dimensional algebras at the end. The focus will be on intuition, visualization and algorithms at the expense of abstraction and generality ("unBourbaki").

Textbook: There will be no required textbook. I will be posting my lecture notes on Owlspace after I complete each module in class. My approach will be most similar to Robert Cahn's book *Semi-simple Lie Algebras and Their Representations*, available on the web at
http://phyweb.lbl.gov/∼rncahn/www/liealgebras/book.html
Possibly useful references include

- *Lie Algebras in Particle Physics*, Howard Georgi
- *Quantum Field Theory*, Michael Peskin and Daniel Schroeder
- *Conformal Field Theory*, Phillipe Di Francesco, Pierre Mathieu, David Sénéchal

(Approximate) Syllabus

1. Review of so(3) and su(2)
    - Rotation group in 2D and nD.

- Generators, commutation relations, ladder operators, fundamental and adjoint representations.
- Irreducible representations, spherical harmonics.
- Spinors.

2. Some elementary Lie group geometry: the 3-sphere [group manifold for SU(2)]
   - Lie group as a manifold. Invariant metric.
   - Left and right generators (Killing vector fields), Hermitian operators on a symmetric space.
3. su(3)
   - Fundamental and adjoint irreps. Cartan subalgebra H.
4. Killing form and commutation relations
   - Root representation in H.
   - Root space scalar product.
   - Commutation relations.
5. Roots and weights
   - Raising and lowering operators, weight chains.
   - Master weight depth formula.
   - Simple roots, Cartan matrix, root geometry, Dynkin diagrams.
6. Classical and exceptional Lie algebras
7. Representations II
   - Root and coroots, marks and comarks.
   - Chevalley basis and Dynkin coefficients. Quadratic form matrix.
   - Complex and real representations.
   - Lattices, conjugacy classes.
   - Tensors and Young tableaux for su(N). Tensors for so(N) and sp(2N).
8. Casimir operators and characters
   - Weyl vector, quadratic Casimir, Freudenthal's formula.
   - Weyl group.
   - Weyl's character formula. Dimension and Strange formulae.
9. Spinor representations of so(2n+1) and so(2n)
   - Orthogonal weights.
   - Spinor irrep., Jordan-Wigner fermions.
   - Clifford algebras.

Additional/alternative topics, interest and time-permitting

1. Riemannian symmetric spaces and random matrix theory; classification of topological phases.
2. Affine lie algebras. Loop group, level, affine roots and dominant weights. 2D CFT and WZNW models. Quantum equivalence.
3. Anyons. Fusion and braiding rules.
4. Quantum groups. Fusion rules for affine Lie algebras.

Learning outcomes

1. Understand the defining relations for simple Lie algebras, and the key components (roots, weights, Weyl group, characters).
2. Master graphical tools and algorithms (weight diagrams, Young tableaux, etc). Tensor decomposition.
3. Understand spinor representations and Clifford algebras.
4. Introduction to more advanced topics.

Grading: Homework 4-6 sets (100%).

Prerequisites: Upper division quantum mechanics and linear algebra. Quantum field theory is not required, although it might be helpful for context.

Any student with a documented disability needing academic adjustments or accommodations is requested to speak with me during the first week of class. Additionally, students will need to contact Disability Support Services in the Allen Center.

Rice Honor Code: In this course, all students will be held to the standards of the Rice Honor Code, a code that you pledged to honor when you matriculated at this institution. If you are unfamiliar with the details of this code and how it is administered, you should consult the Honor System Handbook at http://honor.rice.edu/honor-system-handbook/. This handbook outlines the University's expectations for the integrity of your academic work, the procedures for resolving alleged violations of those expectations, and the rights and responsibilities of students and faculty members throughout the process.