1. Rotations in 3D, so(3), and su(2).
* version 2.0 *

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September 5, 2016

Contents

The discussion here largely follows chapter I of [1]. In addition to serving as a review, the last two sections collect a variety of useful facts and formulae for su(2) = so(3) representations.

1.1 Rotation groups in 3D

1.1.1 $SO(2) \simeq U(1)$

The set of rotations about a given fixed axis in space define an **abelian**, **continuous** group. Consider a 3D vector $V = \{V_x, V_y, V_z\}$. A rotation in the xy plane (around the z axis) is represented by the matrix

$$\hat{R}(\theta) \equiv \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.1.1}$$

An active counterclockwise rotation by the angle θ is implemented by the matrix multiplication

$$V' \equiv \hat{R}(\theta)\, V = \begin{bmatrix} \cos(\theta)V^x - \sin(\theta)V^y \\ \sin(\theta)V^x + \cos(\theta)V^y \\ V^z \end{bmatrix} = \begin{bmatrix} V'^x \\ V'^y \\ V'^z \end{bmatrix}. \tag{1.1.2}$$

A **continuous group** G is a set of transformations satisfying group axioms,[1] in which each element $g(x) \in G$ can be specified by a continuously variable parameter or set of parameters $x = \{x_i\}$. Without loss of generality, we will consider real-valued parameters

[1] A **group** is a set $G = \{g_\alpha\} = \{g_1, g_2, \ldots\}$ along with a binary rule ("multiplication") $G \times G \to G$, satisfying the properties

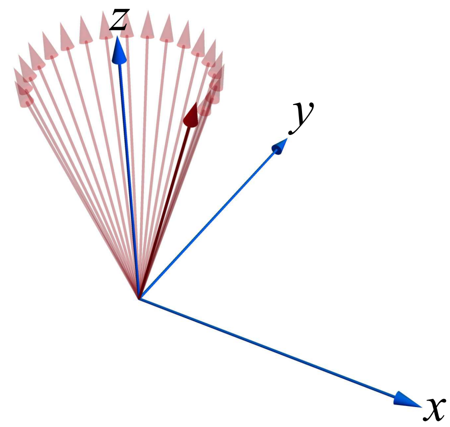

Figure 1.1: Rotation of a 3D vector around the z-axis.

$(x_i \in \mathbb{R}\ \forall\ i)$. Unlike a finite group such as the set S_n of permutations of n objects, a continuous group clearly has an uncountably infinite number of elements. Instead, we can define the **dimension** d of a continuous group as the number of parameters needed to specify all group transformations, i.e. the $\{x_i\} = \{x_1, x_2, \ldots, x_d\}$. We can think of each group element $g(x)$ labeled by $x = \{x_i\}$ as a point on a d-dimensional **group manifold**, a notion we will explore a bit in module 2.

Rotations about a fixed axis have $d = 1$, specified by the angle θ. This group is also **compact**, meaning that we can label any element of the group by restricting θ to a finite interval of the real line: $\theta \in [-\pi, \pi)$. Equivalently, the group manifold is the circle. The inverse of a given rotation $\hat{R}(\theta)$ is a rotation of the same "strength" and opposite chirality:

$$\hat{R}(\theta)\,\hat{R}(-\theta) = \hat{1}_3,$$

where $\hat{1}_3$ is the 3×3 identity matrix.

Continuous symmetry transformations are usually represented by **unitary** operators in quantum mechanics. For a system with a finite n-dimensional Hilbert space, this simply means an $n \times n$ matrix $\hat{U}$ that satisfies

$$\hat{U}\,\hat{U}^\dagger = \hat{U}^\dagger\,\hat{U} = \hat{1}_n, \tag{1.1.3}$$

where the dagger † means the transpose conjugate. Eq. (1.1.2) is a unitary transformation. In addition, the elements of $\hat{R}(\theta)$ are purely real. This means that the inverse is the transpose:

$$\hat{R}^{-1}(\theta) = \hat{R}(-\theta) = \hat{R}^\mathsf{T}(\theta). \tag{1.1.4}$$

A unitary transformation satisfying Eq. (1.1.4) is said to be **orthogonal**. For a more in-depth review of orthogonal transformations in relation to matrix algebra, see e.g. Chapter 4 in [2].

We can write Eq. (1.1.1) in another way. Let us define three 3×3 matrices:

$$\hat{J}_x \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \hat{J}_y \equiv \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \hat{J}_z \equiv \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.1.5}$$

The xy-plane rotation can then be expressed as

$$\hat{R}(\theta) = \exp\left(\theta\,\hat{J}_z\right). \tag{1.1.6}$$

Here the exponential function of a matrix is defined through the expansion

$$e^{\hat{A}} \equiv \hat{1}_3 + \hat{A} + \frac{1}{2!}\hat{A}^2 + \ldots$$

1. There exists an **identity element** $e \in G$ such that
$$g_\alpha \times e = e \times g_\alpha = g_\alpha,$$
2. For any element $g_\alpha \in G$, there exists an **inverse** element g_α^{-1} also in G, such that
$$g_\alpha \times g_\alpha^{-1} = g_\alpha^{-1} \times g_\alpha = e.$$

• **Exercise**: Check Eq. (1.1.6) by expanding in powers of θ and using

$$\hat{J}_z^2 = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly we have

$$\left[\hat{R}(\theta), \hat{R}(\theta')\right] = 0 \tag{1.1.7}$$

for arbitrary θ and θ'. Here $[\hat{A}, \hat{B}] = \hat{A}\,\hat{B} - \hat{B}\,\hat{A}$ denotes the matrix commutator. Eq. (1.1.7) implies that the group of planar rotations is **abelian**: all possible group transformations commute. It means that the order in which a sequence of different rotations is applied to a vector in the plane does not matter:

$$\hat{R}(\theta_1)\,\hat{R}(\theta_2)\,\hat{R}(\theta_3)\,V = \hat{R}(\theta_2)\,\hat{R}(\theta_1)\,\hat{R}(\theta_3)\,V = \hat{R}(\theta_3)\,\hat{R}(\theta_1)\,\hat{R}(\theta_2)\,V = \hat{R}(\theta_1+\theta_2+\theta_3)\,V.$$

The last equality follows from Eq. (1.1.6). Higher dimensional continuous abelian groups are also possible, e.g. the set of all diagonal $n \times n$ matrices.[2]

The abelian group of rotations in a plane is denoted SO(2), meaning the special[3] orthogonal group acting on a vector (or its projection into the plane) in two dimensions. It is also denoted by U(1), the unitary group formed by the composition of complex phase factors; it also called the circle group.

1.1.2 SO(3)

If we consider rotations about *different* axes in 3 or more spatial dimensions, then story is different. To see this, we consider **infinitesimal rotations**, that is, continuous group elements that are arbitrarily close to the group identity. The latter is a unique point on the group manifold, conventionally labeled by the origin of the group parameter coordinate system $\{x^i\} = 0$. Eq. (1.1.6) implies that

$$\hat{R}_{xy}(\theta_3) \simeq \hat{1}_3 + \theta_3\,\hat{J}_z, \quad |\theta_3| \ll 1. \tag{1.1.8}$$

For a right-handed coordinate system in 3D [Fig. 1.1], we can represent infinitesimal yz and zx plane rotations using the other "generators" in Eq. (1.1.5),

$$\hat{R}_{yz}(\theta_1) \simeq \hat{1}_3 + \theta_1\,\hat{J}_x, \quad \hat{R}_{zx}(\theta_2) \simeq \hat{1}_3 + \theta_2\,\hat{J}_y, \quad |\theta_{1,2}| \ll 1. \tag{1.1.9}$$

The corresponding finite yz and zx plane rotations are exponentiations of these.

A rotation within *any* plane (or about any axis perpendicular to it) in 3D can be composed from a product of three finite rotations about at least two different axes; a concrete construction is the Euler angle scheme used in classical mechanics texts [2], but we will not need this here. We will instead focus on products of infinitesimal rotations. Consider a product of yz and zx plane rotations, applied in opposite orders. The difference between these is given by

$$\hat{R}_{yz}(\theta_1)\,\hat{R}_{zx}(\theta_2) - \hat{R}_{zx}(\theta_2)\,\hat{R}_{yz}(\theta_1) \simeq \theta_1\theta_2\left[\hat{J}_x, \hat{J}_y\right]. \tag{1.1.10}$$

Order therefore matters for the composition of rotations about orthogonal axes, with different results for different orders. For infinitesimal transformations, we need to know the **commutation relations** between the **generators** $\hat{J}_{x,y,z}$ of different rotations. For rotations in 3D, these are given by

$$\left[\hat{J}_x, \hat{J}_y\right] = \hat{J}_z, \quad \left[\hat{J}_y, \hat{J}_z\right] = \hat{J}_x, \quad \left[\hat{J}_z, \hat{J}_x\right] = \hat{J}_y. \tag{1.1.11}$$

• **Exercise**: Verify Eq. (1.1.11) using Eq. (1.1.5).

A more succinct statement employs the rank-3 **Levi-Civita** tensor. This is the unique completely antisymmetric third rank tensor with three indices $\{i, j, k\}$ which run over three values $i \in \{1, 2, 3\}$ or $i \in \{x, y, z\}$:

$$\begin{aligned} \epsilon_{ijk} &= -\,\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji} \\ &= \epsilon_{jki} = \epsilon_{kij}. \end{aligned} \tag{1.1.12}$$

[2]The representations of higher dimensional continuous abelian groups are however completely reducible.

[3]Special orthogonal transformations are defined to have determinant one. This includes rotations, but excludes discrete orthogonal transformations with determinant minus one, as well as products of these with rotations. Discrete orthogonal transformations with determinant minus one are reflections and (in odd spatial dimensionality) inversion.

In other words, it is antisymmetric under the exchange of any two indices, and symmetric under cyclic permutations. It is normalized such that

$$\epsilon_{123} = 1.$$

Obviously $\epsilon_{ijk} = 0$ for any two or more indices set equal to each other. Then Eq. (1.1.11) can be written as

$$\left[\hat{J}_i, \hat{J}_j\right] = \epsilon_{ijk}\,\hat{J}_k. \tag{1.1.13}$$

Eq. (1.1.13) specifies the **Lie algebra** associated to the group of rotations in three spatial dimensions. The group is denoted SO(3) (special [3] orthogonal in 3D), and the Lie algebra by so(3). A continuous group generated by a nontrivial Lie algebra (i.e., a Lie algebra with nontrivial commutation relations) is said to be **non-abelian**. The key data is encoded in the **structure constants** or non-vanishing commutation relations. For so(3), these are the components of the Levi-Civita tensor ϵ_{ijk}.

The Lie algebra encodes most aspects of what we want to know about a continuous group. A generic element of the group can be expressed as the exponentiation of a linear combination of generators, with coefficients specified by the group manifold coordinates. E.g., a generic SO(3) rotation can be written as

$$\hat{R}(\boldsymbol{\theta}) \equiv \exp\left(\theta_i\,\hat{J}_i\right), \tag{1.1.14}$$

where we assume Einstein index notation (repeated indices are summed over appropriate values—here $i \in \{1, 2, 3\}$). We can compose two arbitrary rotations using only the commutation relations, via the Baker-Campbell-Hausdorff matrix identity

$$\exp(\hat{A})\,\exp(\hat{B}) = \exp\left\{\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}] + \frac{1}{12}\left(\left[\hat{A}, [\hat{A}, \hat{B}]\right] + \left[\hat{B}, [\hat{B}, \hat{A}]\right]\right) + \ldots\right\} \equiv \exp(\hat{C}). \tag{1.1.15}$$

An explicit solution for $\hat{C}$ in terms of $\hat{A}$ and $\hat{B}$ is equivalent to a global group map that takes any two points on the group manifold, and returns a third. Because the result depends only on the commutator algebra, this map is a property of the group itself, and not the particular representation in which the generators $\{\hat{J}_i\}$ have been defined. In practice, the composition of generic group transformations is complicated, since it requires understanding both local (geometrical) and global (topological) aspects of the group manifold; explicit formulae exist only for simple cases.

In physics we typically don't need to understand group composition in detail. Instead, we usually want to classify and understand different representations of the Lie algebra. The representation in which we have defined the 3×3 matrix generators $\{\hat{J}_i\}$ in Eq. (1.1.5) is called the **vector** or **defining** representation. It is interesting to note that the numerical elements of the three generators are in one-to-one correspondence with those of the Levi-Civita symbol itself, i.e.

$$[J_j]_{ik} = \epsilon_{ijk}. \tag{1.1.16}$$

Thus the structure constants themselves also form a representation, known as the **adjoint representation**. It is the representation to which the elements of the Lie algebra themselves belong, and plays a crucial role in subsequent developments. The algebra so(3) is special because the defining and adjoint representations are the same. This is not true for su(2) (reviewed below) or most other Lie algebras.

In quantum theory, one often wants to classify different objects by symmetry. These can be matrix or differential operators acting on a Hilbert space in one-body quantum mechanics, or composite field operators in a many-particle quantum field theory. The key question then arises: if we have an "elementary" object ϕ_i that transforms in a certain way under a continuous group operation, how can we understand the transformation properties of a "composite" object such as a tensor of m indices, built from ϕ_i:

$$T_{i_1 i_2 \ldots i_m} \equiv \phi_{i_1}\,\phi_{i_2}\,\times \cdots \times \phi_{i_m}.$$

(In this case the tensor T is completely symmetric). Instead of determining the transformation properties of T directly from ϕ, it is better to take a detour to study the classification of so-called **irreducible representations** of the Lie algebra itself. The idea is that the commutation algebra Eq. (1.1.13) and the group composition hold abstractly, independent of representation. Any "composite" object like T should decompose into pieces that transform under different representations of the Lie algebra. These different representations are characterized by certain data such as the angular momentum quantum numbers (reviewed below), and these numbers determine the physical properties of the objects of interest in quantum theory.

1.2 Lie algebra: formal definition

A Lie algebra is technically a vector space L: a collection of objects with notions of addition, scalar multiplication, additive inverse and identity, and completeness. We do *not* impose an inner product structure (yet). In addition, a Lie algebra is equipped with a bilinear operation $[\,,\,]$ ($L \times L \to L$), called a "Lie product," "Lie bracket," or "commutator," that satisfies the conditions

$$[x,y] = -[y,x] = z, \qquad x,y,z \in L \qquad \text{definition, antisymmetry.} \tag{1.2.1a}$$
$$[ax,y] = a[x,y], \qquad x,y \in L, \quad a \in \{\mathbb{R},\mathbb{C},\ldots\} \qquad \text{scalar multiplication.} \tag{1.2.1b}$$
$$[x_1+x_2,y] = [x_1,y]+[x_2,y], \qquad x_1,x_2,y \in L \qquad \text{linearity.} \tag{1.2.1c}$$
$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0, \qquad x,y,z \in L \qquad \text{Jacobi identity.} \tag{1.2.1d}$$

The second and third conditions follow from the idea that if two elements x and y belong to L (a linear vector space), then so does $ax + by$, where a, b are scalar coefficients. The fourth (Jacobi) identity is very important. It holds identically for any representation of Lie algebra elements as $n \times n$ matrices, but we are insisting that this condition generalizes to the abstract definition of the algebra (wherein we are not required to view the Lie product as a matrix commutator).

1.3 su(2) $\simeq$ so(3); irreducible representations

As should be familiar from the theory of spin angular momentum in quantum mechanics, the commutation algebra for the generators of SU(2) transformations on 2-component complex vectors (or "spinors") satisfy the same Lie algebra as so(3). One says that su(2) = so(3), although the corresponding groups SO(3) and SU(2) are not globally equivalent.

1.3.1 Spin 1/2

For spin 1/2, one defines the Hermitian generators[4]

$$\hat{T}_i \equiv \frac{1}{2}\hat{\sigma}_i, \quad \left[\hat{T}_i,\hat{T}_j\right] = i\epsilon_{ijk}\,\hat{T}_k, \tag{1.3.1}$$

where the Pauli matrices are defined in the standard basis,

$$\hat{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \hat{\sigma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.3.2}$$

The spin 1/2 representation is especially tractable because manipulations with Pauli matrices are facilitated by their *anticommutation* relations,

$$\{\hat{\sigma}_i,\hat{\sigma}_j\} = 2\hat{1}_2\,\delta_{i,j}, \tag{1.3.3}$$

where $\{\hat{A},\hat{B}\} \equiv \hat{A}\,\hat{B} + \hat{B}\,\hat{A}$ is the matrix anticommutator. Technically, we can view Eq. (1.3.3) as a **Clifford algebra**. Such an algebra of operators always arises in the **spinor representations** of the orthogonal group so(n) (although the latter is not equivalent to any other Lie algebra for $n > 6$).

For the Lie algebra su(2), spin 1/2 is the defining or **fundamental** representation. By contrast, the su(2) adjoint representation is the defining or 3-vector representation of so(3). Since it is the representation in which the generators themselves transform, the dimensionality of the adjoint is always equal to d, the dimension of the group.

We summarize some properties of the spin 1/2 representation, most of which should already be familiar:

1. A generic spinor state $|\hat{n}\rangle \equiv \alpha\,|{\uparrow}\rangle + \beta\,|{\downarrow}\rangle$ always "points somewhere" on the Bloch sphere:

$$\hat{n}\cdot\hat{\boldsymbol{\sigma}}\,|\hat{n}\rangle = |\hat{n}\rangle\,,$$

where $\hat{n} = \hat{n}(\alpha,\beta)$ is a real unit vector and $\hat{\boldsymbol{\sigma}} \equiv \hat{\sigma}_i\,\hat{i}$ ($i \in \{x,y,z\}$) is the vector of Pauli matrices. The ket $|\hat{n}\rangle$ is a combination of the spin up and down base kets with complex coefficients; in the standard basis [Eq. (1.3.2)]

$$|{\uparrow}\rangle \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |{\downarrow}\rangle \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- **Exercise**: Determine the real unit vector $\hat{n}$ as a function of the complex coefficients $\{\alpha,\beta\}$.

[4]Note that the so(3) generators defined via Eq. (1.1.5) were taken to be real and thus anti-Hermitian, hence the extra factor of i in Eq. (1.3.1) relative to Eq. (1.1.13).

2. A finite SU(2) transformation can be written as

$$e^{-i\frac{1}{2}\boldsymbol{\theta}\cdot\hat{\boldsymbol{\sigma}}} = \hat{1}_2 \cos\left(\frac{\theta}{2}\right) - i\,\hat{\theta}\cdot\hat{\boldsymbol{\sigma}}\sin\left(\frac{\theta}{2}\right), \tag{1.3.4}$$

where $\theta = |\boldsymbol{\theta}|$ is the norm of $\boldsymbol{\theta} = \theta_1\,\hat{1} + \theta_2\,\hat{2} + \theta_3\,\hat{3}$.

3. Some useful Pauli matrix identities

$$\hat{\sigma}_i\,\hat{\sigma}_j = \delta_{i,j}\hat{1}_2 + i\,\epsilon_{ijk}\,\hat{\sigma}_k, \tag{1.3.5a}$$

$$[\hat{\sigma}_i]_{a,b}\,[\hat{\sigma}_i]_{c,d} + \delta_{a,b}\,\delta_{c,d} = 2\delta_{a,d}\,\delta_{c,b}, \qquad \text{SU(2) Fierz identity} \tag{1.3.5b}$$

$$[\hat{\sigma}_i]_{a,b}\,[\hat{\sigma}_i]_{c,d} = \delta_{a,d}\,\delta_{c,b} + [\hat{\sigma}_2]_{a,c}\,[\hat{\sigma}_2]_{b,d}. \qquad \text{Sp(2) Fierz identity} \tag{1.3.5c}$$

In the last two equations, one sums over the three generators on the left-hand side. We will prove Fierz identities for generic Lie algebras later. Here Sp(2n) denotes the **symplectic** Lie group, another "classical" family that we will eventually discuss. The three simplest Lie algebras are all equivalent, so(3) = su(2) = sp(2).

1.3.2 Generic j

We assume the existence of a generic representation in which the su(2) generators $\hat{T}_i$ are $n \times n$ Hermitian matrices. We rewrite Eq. (1.3.1) in terms of raising and lowering operators:

$$\hat{T}_\pm \equiv \hat{T}_x \pm i\hat{T}_y, \qquad \left[\hat{T}_z, \hat{T}_\pm\right] = \pm\hat{T}_\pm, \quad \left[\hat{T}_+, \hat{T}_-\right] = 2\hat{T}_z, \tag{1.3.6}$$

We assume the existence of a **highest weight state** $|j\rangle$ that satisfies

$$\hat{T}_z\,|j\rangle = j\,|j\rangle\,, \quad \hat{T}_+\,|j\rangle = 0, \quad \text{highest weight state.} \tag{1.3.7}$$

Then we consider

$$\hat{T}_z\,\hat{T}_-\,|j\rangle = \left(\hat{T}_-\,\hat{T}_z - \hat{T}_-\right)|j\rangle = (j-1)\,\hat{T}_-\,|j\rangle\,, \tag{1.3.8}$$

so that $\hat{T}_-\,|j\rangle$ is an eigenstate of $\hat{T}_z$ with eigenvalue $(j-1)$. We define

$$|k-1\rangle \equiv \hat{T}_-\,|k\rangle\,, \tag{1.3.9}$$

where $\langle k|k\rangle$ is not generally normalized to one. Then, in order to obtain a <u>finite</u> representation, we must have for some q

$$\hat{T}_-\,|q\rangle = 0, \quad \text{lowest weight state.} \tag{1.3.10}$$

What is q?

$$\hat{T}_z\,\hat{T}_+\,|k\rangle = \left(\hat{T}_+\,\hat{T}_z + \hat{T}_+\right)|k\rangle = (k+1)\hat{T}_+\,|k\rangle\,.$$

Define

$$\hat{T}_+\,|k\rangle \equiv r_k\,|k+1\rangle \tag{1.3.11}$$

so that

$$\hat{T}_+\,\hat{T}_-\,|k+1\rangle = \left(\hat{T}_-\,\hat{T}_+ + 2\hat{T}_z\right)|k+1\rangle = [r_{k+1} + 2(k+1)]\,|k+1\rangle \tag{1.3.12}$$

implying the recursion relation

$$r_{k-1} = r_k + 2k. \tag{1.3.13}$$

Eq. (1.3.7) implies that $r_j = 0$ (highest weight state assumption), so that

$$\begin{aligned}
r_{j-1} &= 2j \\
r_{j-2} &= 2j + 2(j-1) \\
r_{j-3} &= 2j + 2(j-1) + 2(j-2) \\
&\;\vdots \\
r_{j-m} &= 2jm - m(m-1).
\end{aligned}$$

or

$$\Rightarrow r_k = j(j+1) - k(k+1). \tag{1.3.14}$$

Returning to Eq. (1.3.10),

$$\begin{aligned} \hat{T}_+ \hat{T}_- |q\rangle &= \left(\hat{T}_- \hat{T}_+ + 2\hat{T}_z\right) |q\rangle \\ &= (r_q + 2q) |q\rangle = 0, \end{aligned} \tag{1.3.15}$$

so that

$$j(j+1) - q(q+1) + 2q = 0. \tag{1.3.16}$$

This has two possible solutions $q = \{j+1, -j\}$. By assumption $q = j - n$ with n some positive integer. Thus $2j \in \mathbb{Z}_+$ (the set of positive integers), and we recover the usual integer and half-integer representations of su(2): $j = 1/2$ (fundamental), $j = 1$ (adjoint), $j = 3/2$, etc.

1.3.3 Quadratic Casimir

Recall that the squared norm of the vector of angular momentum generators takes a constant value for an irreducible representation of su(2). This is the first example of a **quadratic Casimir operator** that can be used to distinguish representations. The Casimir is

$$\hat{C} \equiv \hat{T}_x^2 + \hat{T}_y^2 + \hat{T}_z^2 = \hat{T}_z^2 + \frac{1}{2}\left(\hat{T}_+ \hat{T}_- + \hat{T}_- \hat{T}_+\right). \tag{1.3.17}$$

Unlike the Lie algebra that exists for abstract generators using the Lie bracket, the Casimir operator can only be defined for a representation: we need to define the ordinary (matrix) product between two generators to write $\hat{C}$ down. Its action on a generic state in the representation defined above is [Using Eq. (1.3.14)]

$$\hat{C} |k\rangle = \left[k^2 + \frac{1}{2}(r_{k-1} + r_k)\right] |k\rangle = j(j+1) |k\rangle . \tag{1.3.18}$$

All states in the representation can be labeled by the eigenvalue $j(j+1)$, since the Casimir commutes with all generators:

$$\left[\hat{C}, \hat{T}_j\right] = \hat{T}_i \left[\hat{T}_i, \hat{T}_j\right] + \left[\hat{T}_i, \hat{T}_j\right] \hat{T}_i = i\epsilon_{ijk} \left(\hat{T}_i \hat{T}_k + \hat{T}_k \hat{T}_i\right) = 0. \tag{1.3.19}$$

1.4 Tensor representations of so(3)

In this section we briefly review integer j representations of su(2) = so(3) in terms of objects that frequently appear in physical calculations.

1.4.1 Traceless symmetric tensors

In the defining $j = 1$ representation of so(3), matrices act on 3-component vectors. We can obtain other representations by tensoring together multiple vector indices. Consider a product of two different vectors V_i and U_j,

$$T_{ij} \equiv U_i V_j .$$

This is a **rank-2** tensor with 9 independent components, and does not transform irreducibly under SO(3) rotations (defined to act identically on both i and j). We can easily decompose T_{ij} into irreducible representations using contractions and (anti)symmetrization:

1. **Trivial** or **scalar** representation, formed from the dot product of two vectors:

   $$\delta^{ij} T_{ij} = T_{ii}, \qquad \text{scalar, } j = 0, \text{ 1 component.} \tag{1.4.1a}$$

   (Einstein summation over i).

2. Vector representation, formed using the Levi-Civita tensor

   $$\epsilon^{ijk} T_{jk} = (\mathbf{U} \times \mathbf{V})^i , \qquad \text{vector, } j = 1, \text{ 3 components.} \tag{1.4.1b}$$

3. Finally there is a traceless symmetric tensor representation,

$$\tilde{T}_{(ij)} \equiv \frac{1}{2}\left(T_{ij} + T_{ji}\right) - \frac{1}{3}\delta_{ij}T_{kk}, \qquad \text{rank-2 tensor, } j = 2, \text{ 5 components.} \tag{1.4.1c}$$

Thus the unsymmetrized tensor decomposes into a direct sum of scalar, vector, and symmetrized, traceless tensor components.[5]

In Eq. (1.4.1c), we have placed the indices in parentheses to indicate symmetrization. More generally, it is useful define complete symmetrization or antisymmetrization operations as follows. Given an arbitrary rank-n tensor $T_{i_1 i_2 \cdots i_n}$,

$$\begin{aligned} T_{(i_1 i_2 \ldots i_n)} &\equiv \frac{1}{n!}\left[T_{i_1 i_2 \cdots i_n} + (\text{permutations})\right], && \text{complete symmetrization,} \\ T_{[i_1 i_2 \ldots i_n]} &\equiv \frac{1}{n!}\left[T_{i_1 i_2 \cdots i_n} + (\text{alternating permutations})\right], && \text{complete antisymmetrization.} \end{aligned} \tag{1.4.2}$$

The representation in Eq. (1.4.1b) [(1.4.1c)] involves $T_{[jk]}$ [$T_{(jk)}$]. The antisymmetric $T_{[jk]}$ is converted into a vector using the Levi-Civita tensor, which is invariant under SO(3) transformations. This is obvious because ϵ^{ijk} has only one independent component, and must therefore transform in the scalar $j = 0$ representation. In Eq. (1.4.1c), the irreducible representation differs from $T_{(jk)}$ by the removal of the scalar trace; this is indicated by the tilde.

We see in the above that the delta function δ^{ij} and the Levi-Civita tensor ϵ^{ijk} play special roles in constructing irreducible representations of so(3). Both are tensors are themselves group invariants, and both can be used to "tie up indices" and create lower rank representations from unsymmetrized higher rank tensors. In generic Lie algebras the Levi-Civita tensor generalizes to a higher rank completely antisymmetric tensor, and this is always a scalar under group transformations. By contrast, the Kronecker delta is special to the orthogonal algebra so(n), and acts as a "metric" for tracing over pairs of n-component vector indices.[6]

All other integer representations of su(2) can be realized in terms of completely symmetrized, traceless tensors. The j representation is associated to a rank-j tensor,

$$\tilde{T}_{(i_1 i_2 \cdots i_j)}; \qquad \tilde{T}_{(kk i_3 \cdots i_j)} = 0. \tag{1.4.3}$$

A completely symmetric tensor with j indices that range over m values has as many independent components as there are distinguishable states that place j bosons into m different states. This is given by

$$\frac{(j + m - 1)!}{j!(m-1)!}.$$

Here $m = 3$; moreover, we must reduce the number of independent components by the number of constraints imposed by tracelessness. Therefore $\tilde{T}_{(i_1 i_2 \cdots i_j)}$ has

$$\frac{(j+2)!}{j!2!} - \frac{(j)!}{(j-2)!2!} = 2j + 1$$

components, as expected.

1.4.2 Spherical tensors

Finally, we recall that one can switch from the symmetrized, traceless Cartesian tensors in Eq. (1.4.3) to spherical tensors $\{\mathcal{O}_{jm}\}$. These are labeled by the representation j and the $\hat{T}_z$ eigenvalue $m \in \{-j, -j+1, \ldots, j-1, j\}$ ("magnetic quantum number"). The components of these are linear combinations of $\tilde{T}_{(i_1 i_2 \cdots i_j)}$ components. An example of spherical tensors are the spherical harmonics $Y_{l,m}$, which can be expressed using Legendre polynomials and the components of a unit vector. I.e., we can write

$$\hat{n} \equiv \{\Pi_x, \Pi_y, \sigma\}, \quad \Pi_x^2 + \Pi_y^2 + \sigma^2 = 1.$$

[5]Note that we are not making a mathematical distinction between "upper" and "lower" indices in the above equations. The integer representations of so(3) are "purely real," meaning there is no notion of a conjugate representation that appears e.g. in su($n \geq 3$). In the latter case, we will use lower (upper) indices to represent the fundamental (conjugate) representations. Here we simply try to mimic this for aesthetics.

[6]For su(n), there is in general no such rank-2 metric that can be used to trace together pairs of fundamental represenation indices. Instead, one must contract one fundamental and one conjugate index. The latter turns out to be equivalent to contracting with the rank-n completely antisymmetric tensor $\epsilon^{i_1 i_2 \cdots i_n}$. This difference between su(n) and so(n) complicates large-n treatments of quantum magnets, since the notions of a "singlet" are different. See the discussion in the next Sec. 1.5.

Defining $\Pi_\pm \equiv \Pi_x \pm i\Pi_y$, one has

$$
\begin{aligned}
Y_{0,0} &= \frac{1}{\sqrt{4\pi}}, & Y_{2,2} &= \frac{1}{\sqrt{4\pi}}\sqrt{\frac{15}{8}}(\Pi_+)^2, \\
& & Y_{2,1} &= -\frac{1}{\sqrt{4\pi}}\sqrt{\frac{15}{2}}\Pi_+\,\sigma, \\
Y_{1,1} &= -\frac{1}{\sqrt{4\pi}}\sqrt{\frac{3}{2}}\Pi_+, & Y_{2,0} &= \frac{1}{\sqrt{4\pi}}\sqrt{5}\left(\frac{3}{2}\sigma^2 - \frac{1}{2}\right), \\
Y_{1,0} &= \frac{1}{\sqrt{4\pi}}\sqrt{3}\sigma, & Y_{2,-1} &= \frac{1}{\sqrt{4\pi}}\sqrt{\frac{15}{2}}\Pi_-\,\sigma, \\
Y_{1,-1} &= \frac{1}{\sqrt{4\pi}}\sqrt{\frac{3}{2}}\Pi_-, & Y_{2,-2} &= \frac{1}{\sqrt{4\pi}}\sqrt{\frac{15}{8}}(\Pi_-)^2,
\end{aligned}
\tag{1.4.4}
$$

etc. Spherical harmonic functions obtain by writing $\Pi_\pm = \sin(\theta)\exp(\pm i\phi)$ and $\sigma = \cos(\theta)$, where θ (ϕ) is the polar (azimuthal) angle, respectively.

1.5 Tensor representations of su(2)

In the previous section, we considered tensors formed by taking a direct product of vector ($j = 1$) representations. A generic rank-n tensor $T_{i_1 i_2 \cdots i_n}$ ($i_j \in \{x, y, z\}$) can be decomposed into irreducible components with $0 \leq j \leq n$; the $j = n$ case is non-degenerate and given by the completely symmetrized, traceless version $\tilde{T}_{(i_1 i_2 \cdots i_n)}$ [Eq. (1.4.3)] (assuming the latter doesn't vanish for the particular T we start with).

We cannot obtain half-integer representations by tensoring together vector indices. Instead, in this section we review tensoring together spin-1/2 (spinor) indices. As should be well-known from the theory of angular momentum addition, one can generate any representation of su(2), integer or half-integer, by tensoring together spinor induces. The fact that any representation can be built from spinor indices generalizes to so(n), as we will later demonstrate.

Imagine spinor wavefunctions ψ_a and ϕ_b, where $a, b \in \{\uparrow, \downarrow\}$ transform in the fundamental spin-1/2 representation of su(2). We can form a second-rank "spinor tensor" (direct product) $T_{ab} \equiv \psi_a \phi_b$. Similar to the second-rank so(3) tensor in Eq. (1.4.1), this can be decomposed into irreducible representations as follows:

1. Scalar representation, formed using the antisymmetric "metric" (rank-2 antisymmetric tensor) $\epsilon^{ab} = -\epsilon^{ba}$ ($\epsilon^{12} = +1$):

$$\epsilon^{ab}\, T_{ab} = \epsilon^{ab}\, T_{[ab]} = T_{\uparrow\downarrow} - T_{\downarrow\uparrow}, \qquad \text{scalar, } j = 0\text{, 1 component (“spin singlet”).} \tag{1.5.1a}$$

   Note that $\epsilon^{ab} = i\,[\hat{\sigma}_2]_{a,b}$.

2. Vector representation

$$T_{(ab)} = \left\{ \begin{array}{cc} T_{\uparrow\uparrow}, & m = 1 \\ \frac{1}{2}\left(T_{\uparrow\downarrow} + T_{\downarrow\uparrow}\right), & m = 0 \\ T_{\downarrow\downarrow}, & m = -1 \end{array} \right., \qquad \text{vector, } j = 1\text{, 3 components (“spin triplet”).} \tag{1.5.1b}$$

   Here m is the $\hat{T}_z$ eigenvalue for the product representation. Of course the spin-1 components listed here (corresponding to kets $\{|\uparrow\uparrow\rangle, \frac{1}{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), |\downarrow\downarrow\rangle\}$ are spherical tensor components of the vector; the Cartesian x, y components are linear combinations of the top and bottom weights.

There is no analogue of the invariant Kronecker delta function for spinor indices [c.f. Eq. (1.4.1a)]. The only invariant tensor that we can use to "tie up" spinor indices is ϵ^{ab}.[7] Thus for a rank-$2n$ spinor tensor $T_{a_1 a_2 \cdots a_{2n-1} a_{2n}}$, we can form a scalar (e.g.) via

$$\epsilon^{a_1 a_2}\epsilon^{a_3 a_4} \times \cdots \times \epsilon^{a_{2n-1} a_{2n}} T_{a_1 a_2 \cdots a_{2n-1} a_{2n}}.$$

In physics language, we are pairing up spin-1/2 indices to form a collection of singlets; for $2n$ spin-(1/2)s, there are many (how many?) ways to do this.

[7] We are ignoring the conjugate representation $\phi^\dagger \to \phi^{*\,a}$, which can be used to form a singlet with ψ_b: $\phi^{*\,a}\psi_a$ (Einstein sum). The main point here is that for su(2) = sp(2), the conjugate and fundamental representations are in fact equivalent, since we can "raise the index" on ψ_b using ϵ^{ab}. We will define conjugate representations for su(n) in due time.

We can form an irreducible representation for arbitrary j by considering a symmetric rank-n tensor of spinor indices:

$$T_{(a_1 a_2 \cdots a_n)}, \quad j = n/2 \text{ representation.} \tag{1.5.2}$$

One way to see that this is irreducible is that the "contraction" of any two indices with ϵ^{ab} is automatically zero. We can also count the number of components, which is

$$\frac{(n+1)!}{n!1!} = n+1,$$

consistent with $n = 2j$.

We will see that the above generalizes to su(n) and sp($2n$), but in different ways. For su(n), the analogue of ϵ^{ab} is the rank-n completely antisymmetric tensor,

$$\epsilon^{a_1 a_2 \cdots a_n}, \quad a_i \in \{1, 2, \ldots, n\}. \tag{1.5.3}$$

This means that one needs n su(n) "spins" (fundamental representation indices) to form a singlet. Such a (rank-$2n$) invariant tensor also exists for sp($2n$). In addition, however, sp($2n$) possesses a generalized $2n \times 2n$ rank-2 antisymmetric tensor

$$\epsilon^{a_1 a_2} \quad \Rightarrow \quad \begin{bmatrix} \hat{0} & \hat{1}_n \\ -\hat{1}_n & \hat{0} \end{bmatrix}, \quad a_i \in \{1, 2, \ldots, 2n-1, 2n\}, \tag{1.5.4}$$

and this can be used to tie up two spinor indices to form a singlet (for any n).

References

[1] Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Benjamin/Cummings, Menlo Park, California, 1984).

[2] Herbert Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Massachusetts, 1980).