2A. SU(n), SO(n), and Sp(2n) Lie groups

* version 1.3 *

Matthew Foster

September 12, 2016

Contents

2A.1 Unitary U(n) and SU(n) ................................................. 1
  2A.1.1 SU(n) tensors .................................................. 2

2A.2 Orthogonal O(n) and SO(n) ......................................... 2
  2A.2.1 SO(n) tensors ................................................. 2

2A.3 Symplectic Sp(2n) .................................................... 3
  2A.3.1 Sp(2n) tensors ................................................. 3

Here we define unitary, orthogonal, and symplectic Lie groups via their fundamental representations. This is a brief “first pass” to acquaint the reader, not a systematic description of these classical Lie groups or their representations. We will return to this subject after developing roots and weights.

2A.1 Unitary U(n) and SU(n)

Consider an \( n \)-component complex vector \( |\psi\rangle \rightarrow [\alpha_1 \alpha_2 \cdots \alpha_n]^T \). Here \( \{\alpha_i\} \in \mathbb{C} \) are the complex components expressed in some basis, and \( T \) denotes the matrix transpose. A generic unitary U(n) transformation is encoded in the \( n \times n \) matrix \( \hat{U} \) via

\[
|\psi\rangle \rightarrow \hat{U} |\psi\rangle,
\]

where

\[
\hat{U}^\dagger \hat{U} = \hat{1}_n.
\] (2A.1.1)

\( \hat{U} \) can be expressed in terms of Hermitian generators \( \{\hat{T}_a\} \) via

\[
\hat{U} = \exp \left( -i \theta^a \hat{T}_a \right), \quad \hat{T}_a^\dagger = \hat{T}_a,
\] (2A.1.2)

where \( \{\theta^a\} \in \mathbb{R}^n \) are the group coordinates. A special unitary SU(n) transformation has determinant one,

\[
\det \hat{U} = 1 \iff \Tr [\hat{T}_a] = 0.
\] (2A.1.3)

In other words, SU(n) has group dimension \( d = n^2 - 1 \), since it excludes the \( n \times n \) identity matrix as a generator. A U(n) transformation can always be written as the product of an abelian U(1) transformation (equivalent to an overall phase) and a non-abelian SU(n) transformation.

---

1Here we have used the formula

\[
\det \hat{A} = \exp(\Tr \log \hat{A}).
\]
2A.1 SU(n) tensors

We will show later that completely antisymmetric and completely symmetric tensors form different classes of irreducible representations:

\[ T^{i_1 i_2 \ldots i_k} : \text{completely antisymmetric rank-} k \text{ tensor} \quad (k \leq n) ; \text{ irreducible rep. of SU}(n) , \]
\[ T^{(i_1 i_2 \ldots i_k)} : \text{completely symmetric rank-} k \text{ tensor} ; \text{ irreducible rep. of SU}(n) . \]

(Mixed symmetry tensors can be classified and decomposed via Young tableaux, as we will eventually see.

2A.2 Orthogonal O(n) and SO(n)

Orthogonal O(n) transformations can be defined as a subgroup of the U(n) transformations in Eq. (2A.1.1). In addition to being unitary, we restrict \( \hat{U}^\dagger = \hat{U}^T \) (real elements).

A special orthogonal transformation SO(n) has determinant one, and excludes reflections, inversion (for \( n \) odd), and products of these with rotations (“improper rotations”). Eq. (2A.2.1) is obviously basis-dependent. We can define orthogonal transformations more generally via

\[ \hat{U}^\dagger \hat{M} \hat{U} = \hat{M} , \quad \hat{M} = \hat{M}^\dagger ; \quad \hat{M}^2 = \hat{1}_n. \quad (2A.2.3) \]

Thus, in addition to being unitary, an O(n) transformation preserves the form of a symmetric “matrix metric” \( \hat{M} \). The SO(n) generators \( \{ \hat{T}_a \} \) satisfy the generalized antisymmetry requirement, Eq. (2A.2.2c). The group dimension \( d = n(n - 1)/2 \) for SO(n), since this is the number of independent matrices that can be chosen to satisfy this constraint.

In Eq. (2A.2.1), \( \hat{M} = \hat{1}_n \). The “metric contraction” with two \( n \)-component vectors \( U_i \) and \( V_j \) is in this case the conventional dot product,

\[ U^\dagger \hat{M} V = U^i M_{ij} V^j = U^i \delta_{ij} V^j . \]

In analogy with differential geometry, we can think of \( M_{ij} \) (here the Kronecker delta) as a covariant metric used to “lower the index” on \( V^j \), so that we can form the inner product \( U^i V_i \). If we make an orthogonal group basis transformation using \( \hat{U} \) that satisfies Eq. (2A.2.1), then obviously the form of \( \hat{M} = \hat{1}_n \) is left invariant. This is not the case if we allow a more general unitary basis transformation, i.e. that generated by a non-orthogonal \( \hat{U} \):

\[ \hat{M} \rightarrow \hat{U}^\dagger \hat{M} \hat{U} \equiv \hat{M}' \neq \hat{M} . \]

Regardless, the symmetry constraint \( \hat{M}'^\dagger = \hat{M}' \) is basis independent.

2A.2.1 SO(n) tensors

Consider a tensor formed from a direct product of fundamental representation (vector) indices. A necessary but not sufficient condition for a rank-\( k \) tensor \( T \) to transform irreducibly under SO(n) is that it is traceless under contraction of any pair of indices with \( M_{ij} \):

\[ M_{p,q} T^{i_1 i_2 \ldots i_k} = 0 \quad \forall \ p,q \in \{ 1, 2, \ldots, k \} \quad (p \neq q) . \quad (2A.2.4) \]

Completely antisymmetric tensors \( T^{[i_1 i_2 \ldots i_k]} \) \( (k \leq n) \) satisfy this condition and indeed correspond to irreducible representations. Traceless completely symmetric tensors \( T^{(i_1 i_2 \ldots i_k)} \) \( [M_{p,q} T^{(i_1 i_2 \ldots i_k)} = 0] \) form a separate class of irreducible representations.
2A.3 Symplectic Sp(2n)

The set of symplectic Sp(2n) transformations form a subgroup of the unitary Lie group U(2n); symplectic transformations and algebras are only defined for even-dimensional spaces. Similar to Eq. (2A.2.2), Sp(2n) transformations can be defined via

\[
\hat{T}_a \hat{\epsilon} = \hat{T}_a, \quad \text{Sp}(2n) \text{ generators},
\]

where the \(2n \times 2n\) matrix \(\hat{\epsilon}\) satisfies

\[
\hat{\epsilon} = -\hat{\epsilon}^T = -\hat{\epsilon}^*; \quad \hat{\epsilon}^2 = -\hat{1}_n.
\]

\[
(2A.3.2)
\]

- The symmetry constraint \(\hat{\epsilon}^T = -\hat{\epsilon}\) is basis independent. Nevertheless, we can always find a basis in which we can view \(\hat{\epsilon}\) as the block \(2n \times 2n\) two-dimensional Levi-Civita tensor [c.f. Eq. (1.5.4)],

\[
\hat{\epsilon} \rightarrow \begin{bmatrix} 0 & \hat{1}_n \\ -\hat{1}_n & 0 \end{bmatrix}.
\]

An alternative notation is to define \(\hat{\sigma}_2 \equiv -i \hat{\epsilon}\), where we view \(\hat{\sigma}_2\) as the block antisymmetric Pauli matrix

\[
\hat{\sigma}_2 \rightarrow \begin{bmatrix} 0 & -i \hat{1}_n \\ i \hat{1}_n & 0 \end{bmatrix}, \quad \hat{U}^T \hat{\sigma}_2 \hat{U} = \hat{\sigma}_2, \quad -\hat{\sigma}_2 \hat{T}_a \hat{\sigma}_2 = \hat{T}_a.
\]

The group dimension \(d = n(2n + 1)\) for Sp(2n), since this is the number of independent matrices that can be chosen to satisfy the constraint. This follows because the symplectic condition is a generalized symmetry constraint on the generators,

\[
\hat{\sigma}_2 \hat{T}_a = \left(\hat{\sigma}_2 \hat{T}_a\right)^T.
\]

The “antisymmetric metric” \(\hat{\epsilon}\) can be used to “tie up” indices transforming in the fundamental representation of Sp(2n). If \(\psi^i\) and \(\phi^j\) are two such representations \((i, j \in \{1, 2, \ldots, 2n\})\), then

\[
\psi^T \hat{\epsilon} \phi = \psi^i \epsilon_{ij} \phi^j
\]

is invariant. The simplest example applies to spin-1/2 spinors in su(2) = sp(2), where this contraction gives the usual singlet state [Eq. (1.5.1a)]. Sp(2n) is different from SU(n) or SU(2n), in that one can always form a group invariant by tensoring together two fundamental rep. indices. Since the fundamental representation of Sp(2n) is in some sense a generalization of spin-1/2, we sometimes refer to it as a “spinor” representation.

2A.3.1 Sp(2n) tensors

Consider a tensor formed from a direct product of fundamental representation indices. A necessary but not sufficient condition for a rank-\(k\) tensor \(T\) to transform irreducibly under Sp(2n) is that it is “traceless” under contraction of any pair of indices with \(\epsilon_{ij}\):

\[
\epsilon_{ip} T^{i_1 i_2 \ldots i_k} = 0 \quad \forall p, q \in \{1, 2, \ldots, k\} \ (p \neq q).
\]

(2A.3.7)

Completely symmetric tensors \(T^{i_1 i_2 \ldots i_k}\) satisfy this condition and indeed correspond to irreducible representations. Traceless completely antisymmetric tensors \(\tilde{T}^{i_1 i_2 \ldots i_k} (k \leq 2n) [\epsilon_{i_1 i_2} \tilde{T}^{i_1 i_2 \ldots i_k}] = 0\) correspond to a different set of irreducible representations.