2A. SU(n), SO(n), and Sp($2n$) Lie groups
* version 1.5 *

Matthew Foster

January 20, 2021

Contents

Here we introduce unitary, orthogonal, and symplectic Lie groups via their defining representations. This is a brief "first pass" to acquaint the reader, not a systematic description of these classical Lie groups or their representations. We will return to this subject after developing roots and weights.

2A.1 Unitary U(n) and SU(n)

Consider an n-component complex vector $|\psi\rangle \to [\alpha_1\, \alpha_2\, \cdots \alpha_n]^{\mathsf{T}}$. Here $\{\alpha_i\} \in \mathbb{C}$ are the complex components expressed in some basis, and T denotes the matrix transpose. A generic unitary U(n) transformation is encoded in the $n \times n$ matrix $\hat{U}$ via

$$|\psi\rangle \to \hat{U}\,|\psi\rangle\,,$$

where

$$\hat{U}^\dagger \hat{U} = \hat{1}_n. \tag{2A.1.1}$$

$\hat{U}$ can be expressed in terms of Hermitian generators $\{\hat{T}_a\}$ via

$$\hat{U} = \exp\left(-i\theta^a\,\hat{T}_a\right), \quad \hat{T}_a^\dagger = \hat{T}_a, \tag{2A.1.2}$$

where $\{\theta^a\} \in \mathbb{R}^n$ are the group coordinates. A special unitary SU(n) transformation has determinant one,[1]

$$\det \hat{U} = 1 \quad \Leftrightarrow \quad \mathsf{Tr}\left[\hat{T}_a\right] = 0. \tag{2A.1.3}$$

In other words, SU(n) has group dimension $d = n^2 - 1$, since it excludes the $n \times n$ identity matrix as a generator. A U(n) transformation can always be written as the product of an abelian U(1) transformation (equivalent to an overall phase) and a non-abelian SU(n) transformation.

[1] Here we have used the formula

$$\det \hat{A} = \exp(\mathsf{Tr} \log \hat{A}).$$

2A.1.1 SU(n) tensors

We will show later that completely antisymmetric and completely symmetric tensors form different classes of irreducible representations:

$$\begin{aligned} T^{[i_1 i_2 \cdots i_k]}: &\quad \text{completely antisymmetric rank-}k\text{ tensor } (k \leq n)\text{; irreducible rep. of SU}(n), \\ T^{(i_1 i_2 \cdots i_k)}: &\quad \text{completely symmetric rank-}k\text{ tensor; irreducible rep. of SU}(n). \end{aligned} \tag{2A.1.4}$$

Mixed symmetry tensors can be classified and decomposed via Young tableaux, as we will eventually see.

2A.2 Orthogonal O(n) and SO(n)

Orthogonal O(n) transformations can be defined as a subgroup of the U(n) transformations in Eq. (2A.1.1). In addition to being unitary, we restrict

$$\hat{U}^\dagger = \hat{U}^\mathsf{T}, \quad \text{(real elements)}. \tag{2A.2.1}$$

A special orthogonal transformation SO(n) has determinant one, and excludes reflections, inversion (for n = odd), and products of these with rotations ("improper rotations").

Eq. (2A.2.1) is obviously basis-dependent. We can define orthogonal transformations more generally via

$$\hat{U}^\dagger \hat{U} = \hat{1}_n \tag{2A.2.2a}$$

$$\hat{U}^\mathsf{T} \hat{M} \hat{U} = \hat{M}, \tag{2A.2.2b}$$

$$-\hat{M}\,\hat{T}_a^\mathsf{T}\,\hat{M} = \hat{T}_a, \quad \text{SO}(n) \text{ generators}, \tag{2A.2.2c}$$

where the $n \times n$ matrix $\hat{M}$ satisfies

$$\hat{M} = \hat{M}^\mathsf{T} = \hat{M}^\dagger; \quad \hat{M}^2 = \hat{1}_n. \tag{2A.2.3}$$

Thus, in addition to being unitary, an O(n) transformation preserves the form of a symmetric "matrix metric" $\hat{M}$. The SO(n) generators $\{\hat{T}_a\}$ satisfy the generalized antisymmetry requirement, Eq. (2A.2.2c). The group dimension $d = n(n-1)/2$ for SO(n), since this is the number of independent matrices that can be chosen to satisfy this constraint.

In Eq. (2A.2.1), $\hat{M} = \hat{1}_n$. The "metric contraction" with two n-component vectors A_i and B_j is in this case the conventional dot product,

$$A^\mathsf{T} \hat{M} B = A^i M_{ij} B^j = A^i \delta_{ij} B^j.$$

In analogy with differential geometry, we can think of M_{ij} (here the Kronecker delta) as a covariant metric used to "lower the index" on B^j, so that we can form the inner product $A^i B_i$. If we make an orthogonal group basis transformation using $\hat{U}$ that satisfies Eq. (2A.2.1), then obviously the form of $\hat{M} = \hat{1}_n$ is left invariant. This is not the case if we allow a more general unitary basis transformation, i.e. that generated by a non-orthogonal $\hat{U}$:

$$\hat{M} \rightarrow \hat{U}^\mathsf{T} \hat{M} \hat{U} \equiv \hat{M}' \neq \hat{M}.$$

• Regardless, the symmetry constraint $\hat{M}'^\mathsf{T} = \hat{M}'$ is basis independent.

2A.2.1 SO(n) tensors

Consider a tensor formed from a direct product of defining representation (vector) indices. A necessary but not sufficient condition for a rank-k tensor T to transform irreducibly under SO(n) is that it is traceless under contraction of any pair of indices with M_{ij}:

$$M_{i_p i_q}\, T^{i_1 i_2 \cdots i_k} = 0 \quad \forall\, p, q \in \{1, 2, \ldots, k\}\ (p \neq q). \tag{2A.2.4}$$

Completely antisymmetric tensors $T^{[i_1 i_2 \cdots i_k]}$ ($k \leq n$) satisfy this condition and indeed correspond to irreducible representations. *Traceless* completely symmetric tensors $\tilde{T}^{(i_1 i_2 \cdots i_k)}$ $[M_{i_1 i_2} \tilde{T}^{(i_1 i_2 \cdots i_k)} = 0]$ form a separate class of irreducible representations.

2A.3 Symplectic Sp($2n$)

The set of symplectic Sp($2n$) transformations form a subgroup of the unitary Lie group U($2n$); symplectic transformations and algebras are only defined for even-dimensional spaces. Similar to Eq. (2A.2.2), Sp($2n$) transformations can be defined via

$$\hat{U}^\dagger \hat{U} = \hat{1}_{2n} \tag{2A.3.1a}$$
$$\hat{U}^\mathsf{T}\, \hat{\epsilon}\, \hat{U} = \hat{\epsilon}, \tag{2A.3.1b}$$
$$\hat{\epsilon}\, \hat{T}_a^\mathsf{T}\, \hat{\epsilon} = \hat{T}_a, \quad \text{Sp}(2n) \text{ generators}, \tag{2A.3.1c}$$

where the $2n \times 2n$ matrix $\hat{\epsilon}$ satisfies

$$\hat{\epsilon} = -\hat{\epsilon}^\mathsf{T} = -\hat{\epsilon}^\dagger; \quad \hat{\epsilon}^2 = -\hat{1}_n. \tag{2A.3.2}$$

• The symmetry constraint $\hat{\epsilon}^\mathsf{T} = -\hat{\epsilon}$ is basis independent. Nevertheless, we can always find a basis in which we can view $\hat{\epsilon}$ as the block $2n \times 2n$ two-dimensional Levi-Civita tensor [c.f. Eq. (1.5.4)],

$$\hat{\epsilon} \rightarrow \begin{bmatrix} 0 & \hat{1}_n \\ -\hat{1}_n & 0 \end{bmatrix}. \tag{2A.3.3}$$

An alternative notation is to define $\hat{\sigma}_2 \equiv -i\,\hat{\epsilon}$, where we view $\hat{\sigma}_2$ as the block antisymmetric Pauli matrix

$$\hat{\sigma}_2 \rightarrow \begin{bmatrix} 0 & -i\,\hat{1}_n \\ i\,\hat{1}_n & 0 \end{bmatrix}, \qquad \begin{aligned} \hat{U}^\mathsf{T}\, \hat{\sigma}^2\, \hat{U} &= \hat{\sigma}^2, \\ -\hat{\sigma}^2\, \hat{T}_a^\mathsf{T}\, \hat{\sigma}^2 &= \hat{T}_a. \end{aligned} \tag{2A.3.4}$$

The group dimension $d = n(2n+1)$ for Sp($2n$), since this is the number of independent matrices that can be chosen to satisfy the constraint. This follows because the symplectic condition is a generalized *symmetry* constraint on the generators,

$$\hat{\sigma}^2\, \hat{T}_a = \left(\hat{\sigma}^2\, \hat{T}_a\right)^\mathsf{T}. \tag{2A.3.5}$$

The "antisymmetric metric" $\hat{\epsilon}$ can be used to "tie up" indices transforming in the defining representation of Sp($2n$). If ψ^i and ϕ^j are two such representations ($i, j \in \{1, 2, \ldots, 2n\}$), then

$$\psi^\mathsf{T}\, \hat{\epsilon}\, \phi = \psi^i\, \epsilon_{ij}\, \phi^j \tag{2A.3.6}$$

is invariant. The simplest example applies to spin-1/2 spinors in su(2) = sp(2), where this contraction gives the usual singlet state [Eq. (1.5.1a)]. Sp($2n$) is different from SU(n) or SU($2n$), in that one can always form a group invariant by tensoring together two defining rep. indices. Since the defining representation of Sp($2n$) is in some sense a generalization of spin-1/2, we sometimes refer to it as a "spinor" representation.

2A.3.1 Sp($2n$) tensors

Consider a tensor formed from a direct product of defining representation indices. A necessary but not sufficient condition for a rank-k tensor T to transform irreducibly under Sp($2n$) is that it is "traceless" under contraction of any pair of indices with ϵ_{ij}:

$$\epsilon_{i_p i_q}\, T^{i_1 i_2 \cdots i_k} = 0 \quad \forall\, p, q \in \{1, 2, \ldots, k\}\ (p \neq q). \tag{2A.3.7}$$

Completely symmetric tensors $T^{(i_1 i_2 \cdots i_k)}$ satisfy this condition and indeed correspond to irreducible representations. *Traceless* completely antisymmetric tensors $\tilde{T}^{[i_1 i_2 \cdots i_k]}$ ($k \leq 2n$) $[\epsilon_{i_1 i_2} \tilde{T}^{[i_1 i_2 \cdots i_k)]} = 0]$ correspond to a different set of irreducible representations.