2. Lie groups as manifolds. SU(2) and the three-sphere.
* version 1.3 *

Matthew Foster

September 5, 2016

Contents

This is the only module in which I will discuss Lie groups and their geometry; subsequent modules will not make use of the results derived here. We will employ some basic notions from differential geometry; an excellent reference is [1]. Introductions to the Haar measure can be found in Refs. [2] and [3].

2.1 The Haar measure

For a Lie group G, each element $g(x)$ is associated to a point on the group manifold, labeled by some coordinates $x = \{x^a\}$, $a \in \{1, 2, \ldots, d\}$. Here d denotes the group dimension. In this module, we follow differential geometry conventions for upper ("contravariant") and lower ("covariant" or dual) indices on tensors. The coordinate x^a is a scalar field defined over the manifold, but its differential variation dx^a is a contravariant vector (or equivalently, a dual basis vector) [1].

A **compact** Lie group has a group manifold is that is closed (has no boundary). The closure of the group under group multiplication implies the existence of a **group map** ϕ: $G \times G \to G$, which takes an arbitrary ordered pair of points (y, x) to a third point z. I.e., if

$$g(z) = g(y) \times g(x) \neq g(x) \times g(y), \tag{2.1.1}$$

then

$$z^a \equiv \phi^a(y; x) \neq \phi^a(x; y), \quad \text{group map.} \tag{2.1.2}$$

The group map $z^a = \phi^a(y; x)$ is an additional global structure defined over the entire manifold. We will see that it implies the existence of a unique integration measure and associated Riemannian metric for the group, tying the abstract structure of the group to a specific manifold geometry.

First, it is helpful to partially "gauge fix" (constrain) the choice of manifold coordinates.

- Let $g(x = 0) = 1$, where "1" denotes the group identity element.
- Let $g(\tilde{x}) \times g(x) = g(x) \times g(\tilde{x}) = 1 \Rightarrow g(\tilde{x}) = g^{-1}(x)$. Thus $\tilde{x}$ locates the inverse of $g(x)$.

Then

$$x^a = \phi^a(x^a; 0) = \phi^a(0; x^a), \qquad \text{multiplication by the group identity.} \tag{2.1.3a}$$

$$(\alpha + \beta)\, x^a = \phi^a(\alpha x; \beta x) = \phi^a(\beta x; \alpha x), \quad \alpha, \beta \in \mathbb{R}, \qquad \text{U(1) composition.} \tag{2.1.3b}$$

$$0 = \phi^a(x, \tilde{x}) = \phi^a(\tilde{x}, x), \qquad \text{group inverse.} \tag{2.1.3c}$$

$$\phi^a(\delta y; \delta x) \simeq \phi^a(\delta x; \delta y) \simeq \delta x^a + \delta y^a + \boldsymbol{O}\left(\delta^2\right), \qquad \text{composition of infinitesimal elements.} \tag{2.1.3d}$$

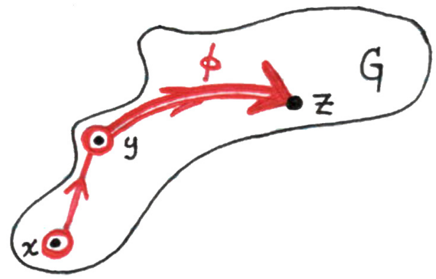

Figure 2.1: Lie group G as a manifold; group map.

In the last equation, we assume that δx and δy are "close" to the group identity 1.

In physical applications like random matrix theory or matrix field theories, one must define an appropriate measure for performing integration over matrix components. In the special case that the matrix represents a group transformation, one may wish to integrate over all possible group transformations. Taking the circle group as an example, we anticipate that a "natural" integration measure is invariant under group transformations. For non-abelian groups, this is called the Haar measure. Let us denote it via

$$d_{\mathsf{H}}[g(x)] \equiv \sqrt{\gamma(x)}\, d^d x, \qquad \gamma(x) = \det\left[\gamma_{ab}(x)\right], \tag{2.1.4}$$

where $\gamma_{ab}(x)$ is the associated **group metric**. We insist that the Haar measure is invariant under **left-** and **right-** group translations:

$$I(y) \equiv \int\limits_G d_{\mathsf{H}}[g(x)]\, f\left[g(y) \times g(x)\right] = \int\limits_G d_{\mathsf{H}}[g(x)]\, f\left[g(x) \times g(y)\right] = \int\limits_G d_{\mathsf{H}}[g(x)]\, f\left[g(x)\right], \tag{2.1.5}$$

where $f(g)$ denotes some arbitrary function of the group element g. Therefore we require that

$$\begin{gathered} I(y) = \int\limits_G d_{\mathsf{H}}[g(x)]\, f\left[g(y) \times g(x)\right] = \int\limits_G d_{\mathsf{H}}[g(\tilde{y}) \times g(z)]\, f\left[g(z)\right] = \int\limits_G d_{\mathsf{H}}[g(z)]\, f\left[g(z)\right], \\ g(z) = g(y) \times g(x), \quad z^a = \phi^a(y; x), \\ g(x) = g(\tilde{y}) \times g(z), \quad x^a = \phi^a(\tilde{y}, z). \end{gathered} \tag{2.1.6}$$

Now changing variables from x to z gives

$$\sqrt{\gamma(x)}\, d^d x = \sqrt{\gamma(x)} \left| \frac{\partial x^a}{\partial z^b} \right| d^d z, \tag{2.1.7}$$

where the $|\cdots|$ is the Jacobian. Comparing this to Eq. (2.1.6), we conclude that[1]

$$\sqrt{\gamma(z)} = \sqrt{\gamma(x)} \left| \frac{\partial \phi^a(\tilde{y}, z)}{\partial z^b} \right|. \tag{2.1.8}$$

We can compute the metric determinant $\gamma(z)$ by setting $x = 0$, so that $g(z) = g(y)$. Furthermore, we know that at the group identity $x = 0$,

$$d_{\mathsf{H}}[g(x \to 0)] = d^d x$$

(up to an arbitrary scale factor), because infinitesimal group transformations commute [c.f. Eq. (2.1.3d)]. We conclude that

$$\sqrt{\gamma(z)} = \left| \frac{\partial \phi^a(\tilde{y}, z)}{\partial z^b} \right|_{y=z}. \tag{2.1.9}$$

[1]Note that Eq. (2.1.8) is *not* merely the transformation law for tensor densities under coordinate diffeomorphisms. The invariant integration measure in that case is

$$\sqrt{\gamma(x)}\, d^d x = \sqrt{h(z)}\, d^d z,$$

where $h = |\det h_{ab}|$ and $h_{ab} = \gamma_{cd}(\partial x^c/\partial z^a)(\partial x^d/\partial z^b)$. The equivalent statement is

$$\sqrt{h(z)} = \sqrt{\gamma(x)} \left| \frac{\partial x^a}{\partial z^b} \right|.$$

For a generic coordinate diffeomorphism, the density $h(z)$ is a *different function* of its arguments from $\gamma(x)$. Invariance of the group integration measure instead constrains $h(z) = \gamma(z)$ on the left-hand side of Eq. (2.1.8).

2.2 The group manifold for SU(2): S_3

We can use the spin 1/2 representation to derive an explicit formula for the group map. Recall the explicit form an SU(2) transformation [Eq. (1.3.4)],

$$\hat{g}(\boldsymbol{\theta}) = \hat{1}_2 \cos\left(\frac{\theta}{2}\right) - i\hat{\theta}\cdot\hat{\boldsymbol{\sigma}} \sin\left(\frac{\theta}{2}\right). \tag{2.2.1}$$

We introduce $\mathbb{R}^4$ (embedding) coordinates

$$x^0 \equiv \cos\left(\frac{\theta}{2}\right), \quad x^i \equiv \sin\left(\frac{\theta}{2}\right)\hat{\theta}^i \quad \Rightarrow \quad \left(x^0\right)^2 = 1 - x^i x^i, \tag{2.2.2}$$

(Einstein summation implied on $i \in \{1,2,3\}$). If we assign a Euclidean metric to the 4D $\{x^0, x^i\}$ space, then the group of SU(2) transformations clearly resides on the unit 3-sphere, denoted S_3. We will see that the Haar measure confirms this.

The group element now reads

$$\hat{g}(x) = \hat{1}_2\, x^0 - i x^i \hat{\sigma}_i. \tag{2.2.3}$$

The group map obtains from

$$\hat{g}(y)\,\hat{g}(x) = \hat{1}_2\,\left(x^0y^0 - x^iy^i\right) - i\hat{\sigma}_i\left(y^0x^i + x^0y^i + \epsilon^{ijk}y^jx^k\right) \equiv g(z), \tag{2.2.4}$$

so that

$$z^0 = \left(x^0y^0 - x^iy^i\right), \quad z^i = \left(y^0x^i + x^0y^i + \epsilon^{ijk}y^jx^k\right). \tag{2.2.5}$$

• **Exercise**: Check that $(z^0)^2 = 1 - z^iz^i$.
Therefore

1. The group map is

$$\phi^i(y;x) = y^0x^i + x^0y^i + \epsilon^{ijk}y^jx^k. \tag{2.2.6}$$

2. In the chosen coordinate system, a group element $g(x)$ is related to its inverse $g(\tilde{x})$ via $\tilde{x}^i = -x^i$.

Therefore

$$\begin{aligned}
\phi^i(-y;z) &= \sqrt{1-y^ky^k}\,z^i - \sqrt{1-z^kz^k}\,y^i - \epsilon^{ijk}y^jz^k, \quad (\text{for } z^0>0),\\
\frac{\partial\phi^i(-y;z)}{\partial z^j} &= \sqrt{1-y^ky^k}\,\delta^i_j + \frac{y^i\,z_j}{\sqrt{1-z^kz^k}} - \epsilon^{ikj}y^k,\\
\Rightarrow \left.\frac{\partial\phi^i(-y;z)}{\partial z^j}\right|_{y=z} &= \sqrt{1-z^kz^k}\,\delta^i_j + \frac{z^i\,z_j}{\sqrt{1-z^kz^k}} + \epsilon^{ijk}z^k.
\end{aligned} \tag{2.2.7}$$

The determinant of this finally gives [via Eq. (2.1.9)]

$$\sqrt{\gamma(z)} = \frac{1}{z^0} = \frac{1}{\sqrt{1-z^iz^i}}. \tag{2.2.8}$$

This is the integration measure for a 3-sphere in terms of constrained Euclidean coordinates, as can be seen from the manifestly invariant integral

$$\int d^4z\;\delta(z^0z^0 + z^iz^i - 1) = \int \frac{d^3z}{\sqrt{1-z^iz^i}}.$$

The group metric can be obtained via the pullback,

$$\gamma_{ij}\,dz^i\,dz^j = \left[\frac{\partial z^0}{\partial z^i}\frac{\partial z^0}{\partial z^j} + \delta_{i,j}\right]dz^i\,dz^j = \left[\delta_{i,j} + \frac{z^i\,z^j}{1-z^kz^k}\right]dz^i\,dz^j. \tag{2.2.9}$$

• **Exercise**: Use Eq. (2.2.9) to verify Eq. (2.2.8). Evaluate the determinant using the formula

$$\det\hat{\gamma} = e^{\mathsf{Tr}\log\hat{\gamma}},$$

writing $\hat{\gamma} = \hat{1}_3 + \delta\hat{\gamma}$, and expanding the log in powers of $\delta\hat{\gamma}$.

2.3 Left- and right- group translations on SU(2): Isometries of S_3

We have shown that the group manifold for SU(2) is S_3. In this section, we demonstrate that continuously generated coordinate transformations that are symmetries of the group manifold ("isometries") encode invariance of the geometry under left- and right-group transformations. To see this, we consider the generators of symmetry transformations on S_3.

Just as S_2 is invariant under SO(3) transformations, S_3 is invariant under SO(4). The generators of SO(4) can be chosen so as to implement SO(2) rotations in each of the orthogonal planes formed by any two coordinate axes in 4D. There are $4 \times 3/2 = 6$ independent planes, and thus 6 generators.[2] We can work out the commutation relations between the generators by thinking about the Hilbert space operators that implement SO(4) rotations in 4D. Recall that in 3D, the z-component angular momentum operator L^z implements rotations in the xy plane and is defined as

$$L^z = -ix\,\partial_y + iy\,\partial_x. \tag{2.3.1}$$

Similarly, we can define the following generators for SO(2) rotations in the 6 orthogonal planes in 4D:

$$\begin{aligned} K^a &= -\,ix^0\partial_a + ix^a\partial_0, && (x^0\text{-}x^a \text{ plane}),\\ J^a &= -\,i\epsilon^{abc}x^b\partial_c, && (x^b\text{-}x^c \text{ plane}), \end{aligned} \tag{2.3.2}$$

$a, b, c \in \{1, 2, 3\}$. The commutation relations are

$$\begin{gathered} \underline{\text{so(4) Lie algebra}}\\ [J^a, J^b] = i\epsilon^{abc}J^c,\\ [J^a, K^b] = i\epsilon^{abc}K^c,\\ [K^a, K^b] = i\epsilon^{abc}J^c. \end{gathered} \tag{2.3.3}$$

Like so(3) = su(2), the so(4) algebra is special, in that it is equivalent to two mutually commuting copies of su(2): so(4) = su(2) $\otimes$ su(2). We will show later that so($2n$+1) [so($2n$)] always has one 2^n dimensional [two 2^{n-1} dimensional] spinor representation(s), in addition to representations formed by tensoring together vector indices. The so(3) and so(4) algebras are sufficiently small that there is essentially nothing else: all representations are be formed by tensoring together spinor su(2) indices for these special algebras. In the case of so(4), the two independent su(2)s can be thought of as "left" and "right" representations, distinguished by chirality. The same decomposition holds for the non-compact version SO(3,1), which governs the Lorentz group of boosts and rotations in 4 spacetime dimensions. The left and right represenations realize left- and right-handed Weyl fermions in 3+1-D relativistic quantum field theory.

We define the linear combinations

$$S^a_\pm \equiv \frac{1}{2}\left(J^a \pm K^a\right). \tag{2.3.4}$$

Then so(4) = su(2) $\otimes$ su(2) follows from

$$[S^a_\sigma, S^b_{\sigma'}] = i\,\delta_{\sigma,\sigma'}\,\epsilon^{abc}S^c_\sigma, \tag{2.3.5}$$

where $\sigma, \sigma' \in \{+, -\}$.

• **Exercise**: Derive Eqs. (2.3.3) and (2.3.5).

We have obtained Eqs. (2.3.3) and (2.3.5) using the representation in Eq. (2.3.2) as differential operators acting in a (flat) four-dimensional Hilbert space. Just as the angular momentum operators in 3D depend only on angular coordinates, these so(4) generators can also be expressed in terms of differential operators acting on the unit 3-sphere, the su(2) group manifold. Technically the generators on S_3 are **Killing vector fields**, and can be derived by either (1) pulling back and raising the 4D dual vectors to the unit sphere, or (2) solving Killing's equation. In a homework problem, you will derive the explicit form of these in some coordinate system on S_3.

Claim: The isometries generated by these commuting su(2)s on the group manifold S_3 correspond to the diffeomorphisms induced by left- and right-multiplying SU(2) group transformations.

Proof: Consider the diffeomorphisms induced by left- and right-su(2) group transformations. Let $y^a = \sin(\theta)\,\hat{y}^a$, $y^0 = \cos(\theta)$.

[2] As we will show in modules 2A and 6, the generators of SO(n) in the defining n-vector representation are in one-to-one correspondence with the set of independent, Hermitian antisymmetric $n \times n$ matrices. There are $n(n-1)/2$ of these.

1. Left group translation: $g(x) \to g(y) \times g(x)$. Then

$$x^a \to \phi^a(y;x) = \cos(\theta)\, x^a + \sin(\theta)\, \hat{y}^a\, x^0 + \sin(\theta)\, \epsilon^{abc}\hat{y}^b x^c. \tag{2.3.6a}$$

2. Right group translation: $g(x) \to g(x) \times g(y)$. Then

$$x^a \to \phi^a(x;y) = \cos(\theta)\, x^a + \sin(\theta)\, \hat{y}^a\, x^0 - \sin(\theta)\, \epsilon^{abc}\hat{y}^b x^c. \tag{2.3.6b}$$

Thus for (e.g.) $\hat{y} = \hat{1}$,

$$\begin{aligned} x^1 &\to \cos(\theta)\, x^1 + \sin(\theta)\, x^0, \\ x^2 &\to \cos(\theta)\, x^2 \mp \sin(\theta)\, x^3, \\ x^3 &\to \cos(\theta)\, x^3 \pm \sin(\theta)\, x^2, \end{aligned} \tag{2.3.7}$$

where the "plus" ("minus") denotes left- (right-)translation.

We compare these to the diffeomorphism induced by the SO(4) rotation

$$x \to \exp\left(\mp i\theta S^1_\pm\right)\, x = \exp\left(\mp i\theta J^1\right)\, \exp\left(-i\theta K^1\right) x. \tag{2.3.8}$$

The first transformation [$\exp\left(-i\theta K^1\right) x$] rotates x in the x^0-x^1 plane, so that

$$\begin{aligned} x^0 &\to \cos(\theta)\, x^0 - \sin(\theta)\, x^1 \\ x^1 &\to \sin(\theta)\, x^0 + \cos(\theta)\, x^1. \end{aligned}$$

The second transformation rotates in the x^2-x^3 plane,

$$\begin{aligned} x^2 &\to \cos(\theta)\, x^2 \mp \sin(\theta)\, x^3, \\ x^3 &\to \pm \sin(\theta)\, x^2 + \cos(\theta)\, x^3. \end{aligned}$$

so that the combination gives exactly Eq. (2.3.7). Q.E.D.

Therefore, we have shown that the so(4) generators that implement symmetry transformations on the three-sphere also implement left and right SU(2) ⊗ SU(2) group translations. The generators (Killing vector fields) $S^a_\pm$ thus form two mutually commuting sets of differential operators that can be used to implement group map transformations across the group manifold. **These conclusions generalize to any Lie group G: there must exist two sets of Killing vector fields that satisfy the associated Lie algebra, but mutually intercommute. Group map transformations are implemented by "exponentiating" linear combinations of these Killing fields.**

Finally, we note that the above manipulations have swept some technicalities under the rug. The coordinates that we are using on the group manifold are $x^{1,2,3}$; $x^0 = \pm\sqrt{1 - x^j x^j}$ is constrained. Therefore the transformations in Eq. (2.3.7) are actually *nonlinear* transformations on these coordinates. A more useful view is to think of $x^{1,2,3}$ as scalar fields defined over the manifold. The explicit form of the differential operators $\{K^a, J^b\}$ ($a, b \in \{1,2,3\}$) expressed solely in terms of $x^{1,2,3}$ will be derived in homework. Using these, one can check that Eq. (2.3.8) does indeed reproduce Eq. (2.3.7).

References

[1] Sean Carroll, *Spacetime and Geometry: An Introduction to General Relativity* (Addison-Wesley 2003); arXiv:gr-qc/9712019.

[2] M. Chaichian and R. Hagedorn, *Symmetries in Quantum Mechanics* (IoP Publishing, Bristol, UK, 1998).

[3] Konstantin Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, Cambridge, England, 1999).