3. First example of a higher rank Lie algebra: su(3)
* version 1.7 *

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Contents

The discussion here largely follows chapter II of [1].

3.1 Gell-Mann, Cartan-Weyl bases

The Lie group SU(3) is defined as the group of determinant one unitary transformations that can be applied to a 3-component, complex vector V (or ket $|V\rangle$ if you prefer). The standard basis for the su(3) generators in the **defining** (or "3") representation in the old physics literature is due to Gell-Mann,

$$\begin{aligned}
&\hat{\lambda}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{\lambda}_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{\lambda}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
&\hat{\lambda}_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \hat{\lambda}_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \\
&\hat{\lambda}_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \hat{\lambda}_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \\
&\qquad\qquad\qquad\qquad \hat{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.
\end{aligned} \tag{3.1.1}$$

We have displayed these in an unconventional way in order to highlight the following points:

- The first row forms an SU(2) subalgebra,

  $$\left[\hat{\lambda}_i, \hat{\lambda}_j\right] = 2i\epsilon_{ijk}\hat{\lambda}_k, \quad i, j, k \in \{1, 2, 3\}.$$

  In this basis, these generators block embed the 2×2 Pauli matrices $\hat{\sigma}_{1,2,3}$ so as to act only on the first 2 components of $|V\rangle$, which transforms as a spin 1/2 under this subalgebra.

- Similarly, $\hat{\lambda}_6$ and $\hat{\lambda}_7$ are embeddings of $\hat{\sigma}_1$ and $\hat{\sigma}_2$ acting on the bottom two components of $|V\rangle$.

- $\hat{\lambda}_4$ and $\hat{\lambda}_5$ are embeddings of $\hat{\sigma}_1$ and $\hat{\sigma}_2$ in the *outer* off-diagonal elements (3,1) and (1,3).

Now, a complete basis for 3×3 matrices has 9 elements, but in su(3) we exclude the identity matrix. This is because the latter merely determines the phase of the determinant for any group transformation. Thus although we have 3 different embeddings of $\hat{\sigma}_1$ and $\hat{\sigma}_2$ amongst $\{\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_6, \hat{\lambda}_7, \hat{\lambda}_4, \hat{\lambda}_5\}$, we have only two diagonal generators $\hat{\lambda}_3$ and $\hat{\lambda}_8$. Note however that we can form the linear combinations

$$\frac{\sqrt{3}}{2}\hat{\lambda}_8 - \frac{1}{2}\hat{\lambda}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \frac{\sqrt{3}}{2}\hat{\lambda}_8 + \frac{1}{2}\hat{\lambda}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{3.1.2}$$

i.e. embeddings of $\hat{\sigma}_3$ that give su(2) subalgebras when combined with $\{\hat{\lambda}_6, \hat{\lambda}_7\}$ and $\{\hat{\lambda}_4, \hat{\lambda}_5\}$, respectively.

We can therefore view the su(3) algebra as a combination of three different su(2) subalgebras. The key point is that these are not independent, and not only because the needed "$\hat{\sigma}_3$ s" are shared between only two diagonal generators $\{\hat{\lambda}_3, \hat{\lambda}_8\}$. The non-trivial structure of su(3) arises because of non-trivial commutation relations between the off-diagonal elements. All representations of su(2) are one-dimensional: there is a highest weight state, and then you use the lowering operator to move down a chain of descendants until you reach the lowest weight state. In su(3) we will see that we can still always define a highest weight state in any irreducible representation, but that

- Acting with only one su(2) lowering operator accesses only a subset (1D chain) of the states (also called **weights**) in the representation, and that
- Acting with different su(2)s allows access to all weights. Representations divide into different classes (**conjugacy classes**), and each conjugacy class forms an infinite **regular lattice** of weights. Different representations in the same class correspond to different finite subsets of the lattice.

All irreducible representations of a given Lie algebra reside on lattices with a common spatial dimensionality r; r is the **rank** of the algebra. su(n) has rank $n-1$, so all representations of su(3) can be described by weights residing on lattices in the plane. A major goal of this course is to identify all possible representations of a given algebra, to develop a geometrical picture of any such representation as a section of a weight lattice, and to acquire tools to obtain the explicit weights and weight multiplicities (degeneracies).

As we will discuss in more detail below, the rank r is equal to the maximum number of independent commuting generators. The set of such generators is the **Cartan subalgebra**, and the weights of a given representation can be labeled by the *mutual* eigenvalues of these. In the Gell-Mann basis for su(3) it is spanned by $\hat{\lambda}_3$ and $\hat{\lambda}_8$,

$$\left[\hat{\lambda}_3, \hat{\lambda}_8\right] = 0. \tag{3.1.3}$$

It is useful to switch from the Gell-Mann to the **Cartan-Weyl** basis.[1] In the latter, we express all off-diagonal generators in terms of raising and lowering operators. We define

$$\begin{aligned} \hat{T}_\pm &\equiv \frac{1}{2}\left(\hat{\lambda}_1 \pm i\hat{\lambda}_2\right), \quad \hat{T}_z \equiv \frac{1}{2}\hat{\lambda}_3, \\ \hat{U}_\pm &\equiv \frac{1}{2}\left(\hat{\lambda}_6 \pm i\hat{\lambda}_7\right), \\ \hat{V}_\pm &\equiv \frac{1}{2}\left(\hat{\lambda}_4 \pm i\hat{\lambda}_5\right), \\ Y &\equiv \frac{1}{\sqrt{3}}\hat{\lambda}_8. \end{aligned} \tag{3.1.4}$$

We associate these 3×3 matrices to abstract Lie algebra generators $\{t_+, t_-, u_+, u_-, v_+, v-, t_z, y\}$. We can determine the structure constants (non-trivial Lie brackets) for the latter by computing all possible matrix commutators between the elements in Eq. (3.1.4). Via Eq. (3.1.2), the brackets between raising and lowering operators in the three su(2) subalgebras are given by

$$\begin{aligned} [t_+, t_-] &= 2t_z, \\ [u_+, u_-] &= 2\left(\tfrac{3}{4}y - \tfrac{1}{2}t_z\right), \\ [v_+, v_-] &= 2\left(\tfrac{3}{4}y + \tfrac{1}{2}t_z\right). \end{aligned} \tag{3.1.5}$$

The only non-trivial Lie bracket between raising operators is

$$[t_+, u_+] = v_+. \tag{3.1.6}$$

[1] We are following Cahn [1] in terms of our naming scheme for raising and lower operators. Later we will switch to the conventional root vector notation, e.g. $t_\pm \to e_{\pm\alpha_1}$, etc.

There are several non-trivial brackets between different types of raising and lowering operators,

$$\begin{aligned}[t_+, v_-] &= -u_-, \\ [u_+, v_-] &= t_-,\end{aligned} \tag{3.1.7}$$

as well as the "Hermitian conjugate" relations (defined with respect to the matrices of the defining representation)

$$\begin{aligned}[v_+, t_-] &= -u_+, \\ [v_+, u_-] &= t_+.\end{aligned}$$

Finally, the Lie brackets between the Cartan subalgebra elements $\{t_z, y\}$ and the raising and lowering operators $\{t_\pm, u_\pm, v_\pm\}$ are given by

$$\begin{aligned}[t_z, t_\pm] &= \pm t_\pm, & [y, t_\pm] &= 0, \\ [t_z, u_\pm] &= \mp \tfrac{1}{2} u_\pm, & [y, u_\pm] &= \pm u_\pm, \\ [t_z, v_\pm] &= \pm \tfrac{1}{2} v_\pm, & [y, v_\pm] &= \pm v_\pm.\end{aligned} \tag{3.1.8}$$

• **Exercise**: Verify Eqs. (3.1.6)–(3.1.8).

3.2 The adjoint representation

The generators of a Lie algebra transform in the **adjoint** representation. This should already be familiar from the notion of a basis or symmetry ("similarity") transformation in quantum mechanics, where an $n \times n$ matrix $\hat{M}$ transforms according to

$$\hat{M} \to \hat{M}' \equiv \hat{U}^\dagger \hat{M} \hat{U}, \qquad \hat{U} \equiv \exp(i\alpha^a \hat{T}_a). \tag{3.2.1}$$

Here $\hat{T}_a$ is an $n \times n$ Hermitian matrix [i.e., a generator of su(n)], and the coordinates $\{\alpha^a \in \mathbb{R}\}$ specify the transformation $\hat{U}$. Since the matrices $\{\hat{T}_a\}$ plus the identity form a basis for $n \times n$ matrices, Eq. (3.2.1) can equivalently be interpreted as a transformation law for the generators themselves,

$$\begin{aligned}\hat{T}_a \to \hat{T}'_a &\equiv \exp(-i\alpha^b \hat{T}_b) \hat{T}_a \exp(i\alpha^c \hat{T}_c) \\ &= \hat{T}_a + i\alpha^b \left[\hat{T}_a, \hat{T}_b\right] + \frac{1}{2!}(i\alpha^b)(i\alpha^c) \left[\left[\hat{T}_a, \hat{T}_b\right], \hat{T}_c\right] + \frac{1}{3!}(i\alpha^b)(i\alpha^c)(i\alpha^d) \left[\left[\left[\hat{T}_a, \hat{T}_b\right], \hat{T}_c\right], \hat{T}_d\right] + \ldots,\end{aligned} \tag{3.2.2}$$

where we have used the Baker-Campbell-Hausdorff formula on the second line. The latter implies that the similarity transformation can be assumed for the abstract Lie algebra L, since the corresponding t'_a is expressed soley in terms of linear combinations of nested Lie brackets.

Consider the transformation on a basis element $x_j \in L$:

$$\mathit{ad}_i(x_j) \equiv [x_i, x_j] = i f_{ijk}\, x_k, \tag{3.2.3}$$

where the f_{ijk} are the structure constants (and the big i is the square root of minus one). For a Lie algebra with dimension d, the operator ad_i can be represented by a $d \times d$ matrix $\widehat{\mathit{ad}}_i$. The argument (x_j) on the left-hand side of Eq. (3.2.3) specifies the j^{th} column of $\widehat{\mathit{ad}}_i$, while the row entries are the structure constants $i f_{ijk}$ for each x_k:

$$\left[\widehat{\mathit{ad}}_i\right]_{k,j} = i f_{ijk}. \tag{3.2.4}$$

We can show that ad_x satisfies the Lie algebra L. Assume that $[x, y] = z$ for some $x, y, z, \in L$. Then we want to see that $[\mathit{ad}_x, \mathit{ad}_y] = \mathit{ad}_z$. Examine the action of these operators on some other element $w \in L$:

$$\begin{aligned}[\mathit{ad}_x, \mathit{ad}_y] w &= \mathit{ad}_x \left[\mathit{ad}_y(w)\right] - \mathit{ad}_y \left[\mathit{ad}_x(w)\right] = [x, [y, w]] - [y, [x, w]] = [x, [y, w]] + [y, [w, x]] \\ &= -[w, [x, y]] = [z, w] = \mathit{ad}_z(w).\end{aligned} \tag{3.2.5}$$

On the second line, we have used the Jacobi identity [Eq. (1.2.1d)].

In the case of su(3), the adjoint representation is 8-dimensional. We associate each of $\{t_+, t_-, u_+, u_-, v_+, v-, t_z, y\}$ to the basis vectors $\{|1\rangle, |2\rangle, \ldots, |8\rangle\}$,

$$a\, t_+ + b\, t_- + c\, u_+ + d\, u_- + e\, v_+ + f\, v_- + g\, t_z + h\, y \to \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix}.$$

Then (e.g.)

$$\widehat{ad}_{t_+} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & t_+ \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_- \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_+ \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & u_- \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & v_+ \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_- \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & t_z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y \\ t_+ & t_- & u_+ & u_- & v_+ & v_- & t_z & y & \end{bmatrix}. \tag{3.2.6}$$

• **Exercise**: Find the matrix representation for $\widehat{ad}_{v_-}$.

3.2.1 Cartan subalgebra H and root H^* spaces

In su(3), the abelian subalgebra t_z and y form a basis for the **Cartan subalgebra $\boldsymbol{H}$**. Any linear combination of these is an element of H (a linear vector space). Consider the adjoint representation of a generic Cartan subalgebra element $a\, t_z + b\, y$. Eq. (3.1.8) implies that the adjoint representation is purely diagonal,

$$\widehat{ad}_{a\, t_z + b\, y} \rightarrow \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_+ \\ 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 & t_- \\ 0 & 0 & -\frac{a}{2}+b & 0 & 0 & 0 & 0 & 0 & u_+ \\ 0 & 0 & 0 & \frac{a}{2}-b & 0 & 0 & 0 & 0 & u_- \\ 0 & 0 & 0 & 0 & \frac{a}{2}+b & 0 & 0 & 0 & v_+ \\ 0 & 0 & 0 & 0 & 0 & -\frac{a}{2}-b & 0 & 0 & v_- \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y \\ t_+ & t_- & u_+ & u_- & v_+ & v_- & t_z & y & \end{bmatrix}. \tag{3.2.7}$$

This is a defining attribute of the Cartan-Weyl basis for L: raising and lowering operators are eigenvectors of a generic Cartan subalgebra element $h \in H$ in the adjoint representation. In su(3), this forces us to choose $\{t_\pm, u_\pm, v_\pm\}$ for the basis elements in L outside of H. By contrast, there is no unique basis for H, since while there are three pairs of raising and lowering operators, at most two "$\hat{\sigma}_3$s" can be chosen independently [see the discussion following Eq. (3.1.2), above].

For any Cartan-Weyl basis element $x_b \in L$ and a generic element of the Cartan subalgebra $h \in H$, we have

$$ad_h(x_b) = \alpha_b(h)\, x_b. \tag{3.2.8}$$

If $x_b \in H$, then $\alpha_b(h) = 0$. Otherwise x_b is a raising or lowering operator. In this case,

1. The eigenvalue α_b is called a **root**, and it is a linear functional on the elements of the Cartan subalgbra.
2. The associated element $x_b \in L$ is called a **root vector**.

It is helpful to think of the space of roots as being dual to the Cartan subalgebra. More precisely, we define $\boldsymbol{H^*}$: The space of linear functionals on H. This is a dual vector space, elements of which are linear combinations of the roots $\{\alpha_b\}$. Eq. (3.2.7) implies that the roots of su(3) can be represented via

$$\begin{aligned} ad_{a\, t_z+b\, y}(t_\pm) &= \alpha_{t_\pm}(a\, t_z + b\, y)\, t_\pm &&= \pm a\, t_\pm, \\ ad_{a\, t_z+b\, y}(u_\pm) &= \alpha_{u_\pm}(a\, t_z + b\, y)\, u_\pm &&= \mp\left(\frac{a}{2} - b\right) u_\pm, \\ ad_{a\, t_z+b\, y}(v_\pm) &= \alpha_{v_\pm}(a\, t_z + b\, y)\, v_\pm &&= \pm\left(\frac{a}{2} + b\right) v_\pm. \end{aligned} \tag{3.2.9}$$

The number of basis elements in L is d, the dimension of the group. The Cartan subalgebra and dual root spaces have dimension r, where $r < d$ is the rank. Although $d = 8$ for su(3), the rank $r = 2$ is much smaller. All irreducible representations can be represented by intersecting weight chains drawn in a 2D plane. The presence of three roots suggests that weights arrange into strings oriented along three different directions in the plane, and this naturally suggests weight lattices of triangular symmetry, as we will soon confirm. It is remarkable that the geometry of the root system translates into precise rules that map out all states in a given irreducible representation. In order to see this connection, we must introduce an inner product on the root space H^*.

References

[1] Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Benjamin/Cummings, Menlo Park, California, 1984).