4. Killing form, root space inner product, and commutation relations
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The discussion here largely follows chapters III and IV of [1].

4.1 Weights in the Cartan-Weyl basis; rank-r bases for H and H^*

As introduced in module 3 with the example of su(3), a Lie algebra L of dimension d expressed in the Cartan-Weyl basis consists of raising and lowering operators (**root vectors**), coupled with a rank $r < d$ abelian Cartan subalgebra $H \in L$. Recall that a **root** is the eigenvalue associated to the action of a generic element $h \in H$ on a root vector in the adjoint representation:

$$ad_h(x_b) = [h, x_b] = \alpha_b(h)\, x_b. \tag{4.1.1}$$

Here x_b is the root vector and α_b is the root, a linear functional on H. The roots belong to the dual vector space H^*. Both H and H^* are linear vector spaces with dimension r.

A generic element ρ of H^* is a linear combination of roots with complex coefficients, referred to as a **weight**; H^* is also referred to as the **space of weights**. The action of ρ on a generic element $k \in H$ is determined by its expansion in terms of roots:

$$\rho \equiv \sum_b c_b\, \alpha_b, \quad \{c_b \in \mathbb{C}\}, \tag{4.1.2}$$

$$\rho(h) = \sum_b c_b\, \alpha_b(h). \tag{4.1.3}$$

Roots refer to specific elements of H^* that correspond to the root vectors in the Cartan-Weyl basis. The total number of root vectors is $d - r$, and the number of these that correspond to "raising operators" (called **positive root vectors**) is half this value. It will turn out the number of positive root vectors is always greater than the rank r, except for the case of su(2) = so(3) = sp(2) (for which they are equal).

For example, in su(3) we have three positive root vectors $\{t_+, u_+, v_+\}$; $r = 2$ and so $d = 3 + 3 + 2 = 8$. A generic Lie group transformation requires the specification of d parameters; in this sense all elements of L (including all positive root vectors) are independent. However, since the number of positive roots is greater than r, it is clear that the roots form an *overcomplete* basis in H^* (thinking of the latter as a linear vector space). If we view the positive roots as "vectors with arrows" pointing somewhere in the r-dimensional H^*, then overcompleteness means that we cannot choose all vectors to be "orthogonal."

Our goal is to bring out the structure of H^* in terms of root geometry. A key tool is the definition of a weight inner product. To define an inner product on H^*, we must first link it with H.

4.2 The Killing form

The first step is to define the **Killing form**, which is the trace normalization for generators in the adjoint representation. Let L be a Lie algebra and $a, b \in L$. Then

$$(a, b) \equiv \mathsf{Tr}\left[\widehat{ad}_a\, \widehat{ad}_b\right], \quad \text{Killing form } L \times L \to \mathbb{C}. \tag{4.2.1}$$

The Killing form itself is *not* a conventional inner product, since there is no requirement that (a, a) be positive definite.[1] If we assume that all $\{\widehat{ad}_a\}$ are Hermitian, then $(a, a) \geq 0$, but we are not required to make this choice.

How do we evaluate the Killing form?

1. We could construct the explicit matrix representations for all adjoint operators in some chosen basis for the Lie algebra, $\{x_i\} \in L \Rightarrow \{\widehat{ad}_i\}$, and then compute the traces of all bilinear products.
2. An easier way is to note that $ad_a\, ad_b(x) = [a, [b, x]]$. Applied to basis elements $\{x_i\}$,
$$ad_i\, ad_j(x_k) = [x_i, [x_j, x_k]] = \sum_l c^{(l)}_{ijk}\, x_l, \tag{4.2.2}$$
where $c^{(k)}_{ijk}$ is the contribution to the trace from the k^{th} basis element. (Recall that in the adjoint representation, e.g.
$$x_1 \to |x_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad x_1 \to \langle x_1| = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$
acting in the trace on the right and left, respectively.)

4.2.1 su(3)

Using the commutation algebra in Eqs. (3.1.5)–(3.1.8) and Eq. (4.2.2), we can determine all non-vanishing Killing forms in the Cartan-Weyl basis for su(3). For example,

$$\begin{aligned}
[t_z, [t_z, t_\pm]] &= t_\pm & &\Rightarrow c^{(t_\pm)}_{t_z t_z t_\pm} = 1, \\
[t_z, [t_z, u_\pm]] &= \frac{1}{4} u_\pm & &\Rightarrow c^{(u_\pm)}_{t_z t_z u_\pm} = \frac{1}{4}, \\
[t_z, [t_z, v_\pm]] &= \frac{1}{4} v_\pm & &\Rightarrow c^{(v_\pm)}_{t_z t_z v_\pm} = \frac{1}{4}, \\
[t_z, [t_z, t_z]] &= 0 & &\Rightarrow c^{(t_z)}_{t_z t_z t_z} = 0, \\
[t_z, [t_z, y]] &= 0 & &\Rightarrow c^{(y)}_{t_z t_z y} = 0,
\end{aligned} \tag{4.2.3}$$

so that $(t_z, t_z) = 3$. The non-vanishing Killing forms in su(3) are

$$\begin{aligned}
(t_z, t_z) &= 3, \\
(y, y) &= 4, \\
(t_+, t_-) &= 6, \\
(u_+, u_-) &= 6, \\
(v_+, v_-) &= 6.
\end{aligned} \tag{4.2.4}$$

• **Exercise**: Verify Eq. (4.2.4).

The Killing forms in Eq. (4.2.4) do not give any hint of the relationship between the different su(2)s, since products involving different "species" of root vectors vanish, e.g. $(t_{\sigma_1}, u_{\sigma_2}) = 0$ for $\sigma_{1,2} \in \{+, -\}$. This result generalizes to all Lie algebras (see below).

[1] Note that Eq. (4.2.1) involves a product of $\widehat{ad}_a$ and $\widehat{ad}_b$, not $(\widehat{ad}_a)^\dagger$ and $\widehat{ad}_b$.

4.3 Root representation in the Cartan subalgebra

Now we establish a link between the Cartan subalgebra H and the dual root space H^*:
Lemma: For any weight ρ in H^*, there exists a unique element h_ρ in H, such that for every $k \in H$,

$$\rho(k) = (h_\rho, k). \tag{4.3.1}$$

In words, we can identify an element of the Cartan subalgebra $h_\rho \in H$, such that the action of the weight $\rho \in H^*$ on any element $k \in H$ is given by the Killing form of h_ρ and k. Thus it relates the action of a dual vector ρ on a vector k to the Killing form between two vectors h_ρ and k. This Lemma is only guaranteed to work for semi-simple Lie algebras, defined precisely in Sec. 4.3.2, below. The requirement is that the "overlap matrix" (h_i, h_j) for any basis in H is invertible, and this is true for semi-simple algebras.

• su(3) example: From Eq. (3.2.9), the positive roots can be taken to be

$$\begin{aligned} \alpha_1\,(a\,t_z + b\,y) &= a, \\ \alpha_2\,(a\,t_z + b\,y) &= -\frac{a}{2} + b, \\ \alpha_3\,(a\,t_z + b\,y) &= \frac{a}{2} + b, \end{aligned} \tag{4.3.2}$$

where we have relabeled $\{t_+, u_+, v_+\} \to \{1, 2, 3\}$. Define

$$h_{\alpha_i} \equiv c_i\,t_z + d_i\,y. \tag{4.3.3}$$

Eq. (4.2.4) implies that

$$(h_{\alpha_i}, t_z) = 3c_i, \quad (h_{\alpha_i}, y) = 4d_i. \tag{4.3.4}$$

Asserting that Eq. (4.3.1) reproduces Eq. (4.3.2) for $\rho = \alpha_{1,2,3}$, we arrive at

$$h_{\alpha_1} = \frac{1}{3}t_z, \quad h_{\alpha_2} = -\frac{1}{6}t_z + \frac{1}{4}y, \quad h_{\alpha_3} = \frac{1}{6}t_z + \frac{1}{4}y. \tag{4.3.5}$$

4.3.1 Weight inner products and root geometry

Define H_0^* to be the space of weights formed from real linear combinations of the roots. This is a subspace of H^* (root combinations with complex coefficients). Assume that α and β are weights in H_0^*. Let $h_{\alpha,\beta}$ be the associated elements in H. We define the **weight inner product** between α and β via

$$\boxed{\langle \alpha, \beta \rangle = (h_\alpha, h_\beta), \quad \text{weight inner product } H_0^* \times H_0^* \to \mathbb{R}.} \tag{4.3.6}$$

Why do we introduce an inner product structure only for elements of H_0^*?

- We want something that gives us geometrical information about the overcomplete basis of roots; as demonstrated in Eq. (4.2.4), the Killing form of the Cartan-Weyl basis elements does not tell us this.
- There are no "naturally distinguished" elements of the Cartan subalgebra H.

The roots are eigenvalue functionals that are independent of the basis chosen for H. Eq. (4.3.1) translates these into special elements of H, and the inner product Eq. (4.3.6) extracts the geometrical root data. Eq. (4.3.6) will enable us to map out irreducible representations by calculating the depth of weight chains generated by lowering with a given root vector (module 5).

In su(3), Eqs. (4.2.4) and (4.3.5) give

$$\begin{aligned} \langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = \langle \alpha_3, \alpha_3 \rangle &= 1/3, \\ \langle \alpha_1, \alpha_2 \rangle &= -1/6, \\ \langle \alpha_1, \alpha_3 \rangle = \langle \alpha_2, \alpha_3 \rangle &= 1/6. \end{aligned} \tag{4.3.7}$$

These results imply that we can visualize $\alpha_{1,2,3}$ as three equal-length vectors in the plane, with $\pi/3$ angles between $\alpha_{1,2}$ and α_3, and a $2\pi/3$ angle between α_1 and α_2. Incorporating the negative roots, we obtain the picture in Fig. 4.1.

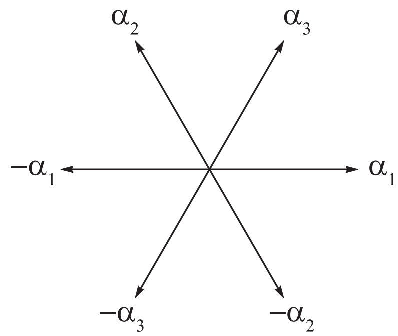

Figure 4.1: Root geometry in su(3), as implied by the weight inner products in Eq. (4.3.7).

4.3.2 Some definitions

1. A **subalgebra** is a subspace M of L that is closed under the Lie bracket.

2. An **ideal** is a special subalgebra J of L, such that if $x \in J$ and $y \in L$, then $[x, y] \in J$.
Example: The generators of spatial translations in a Hilbert space (momentum operators) $[\hat{P}_i, \hat{P}_j] = 0$ form an abelian, ideal subalgebra of the 3D Euclidean "Poincaré" algebra. The latter consists of translation and rotation generators,

$$\begin{aligned}
[\hat{P}_i, \hat{P}_j] &= 0, \\
[\hat{L}_i, \hat{P}_j] &= i\epsilon_{ijk}\hat{P}_k, \\
[\hat{L}_i, \hat{L}_j] &= i\epsilon_{ijk}\hat{L}_k.
\end{aligned} \tag{4.3.8}$$

Example: consider $u(n) = u(1) \otimes \mathrm{su}(n)$, the generators of $n \times n$ unitary matrices. The u(1) generator $\hat{1}_n$ is an abelian ideal. The set of traceless Hermitian su(n) generators is a non-abelian ideal.

3. A **simple** Lie algebra has no ideals except the full algebra and 0 (null).

4. A **semi-simple** Lie algebra has no abelian ideals. Thus neither the Poincaré nor u(n) algebras are semi-simple, while su(3) is simple. In general, a semi-simple algebra is a sum of simple non-abelian ideals.

4.4 The structure of simple Lie algebras

4.4.1 Root facts

Our exploration of su(3) in module 3 suggested that a Lie algebra L of dimension d can be partitioned in the Cartan-Weyl basis via the Lie bracket relations with a generic Cartan subalgebra element:

- **Cartan subalgebra H basis.** This consists of r independent elements $\{h_i\}$, $i \in \{1, \ldots, r\}$; r is the rank. The $\{h_i\}$ mutually commute,

$$ad_{h_i}(h_j) = [h_i, h_j] = 0. \tag{4.4.1}$$

- **Root vectors.** A root vector e_α satisfies [Eq. (4.1.1)]

$$ad_h(e_\alpha) = [h, e_\alpha] = \alpha(h)\, e_\alpha, \quad h \in H, \tag{4.4.2}$$

where $\alpha \in H^*$ is a root (a functional on the Cartan subalgebra H). Eq. (4.4.2) is the analog of the su(2) equation $[s^z, s^\pm] = (\pm 1)s^\pm$, where s^z is the sole Cartan subalgebra element, and $s^\pm$ denote positive and negative root vectors, i.e. raising and lowering operators when acting on some representation.

In the case of su(2), raising and lowering is by one unit. For a generic Lie algebra, Eq. (4.4.2) implies that one moves up and down weight chains via increments determined by the roots.

We assert the following facts about roots, some of which we will prove later:

1. If α (e_α) is a root (root vector), then $-\alpha$ ($e_{-\alpha}$) is also a root (root vector). Because α is a functional and not a number, there is no unique way to define "raising" and "lowering," but we can always make some choice to regard (e.g.) e_α ($e_{-\alpha}$) as the raising (lowering) operator in a string of weights appearing in some irreducible representation. In su(3), we found three different root vector pairs $\{t_\pm\}, \{u_\pm\}, \{v_\pm\}$, c.f. Eq. (3.1.8); the associated roots were respectively relabeled $\pm\alpha_{1,2,3}$ in Eqs. (4.3.2) and (4.3.7).

2. Each root α_i uniquely corresponds to a particular root vector e_{α_i}. In other words, the roots are non-degenerate.

3. If α is a root, then 2α is not a root. It means that α and $-\alpha$ are highest and lowest weight states in L itself, if we act on either with $e_{\pm\alpha}$. Along with a middle element with weight 0 (corresponding to some particular $h_\alpha \in H$, defined below), the weight chain $\{-\alpha, 0, \alpha\}$ is the adjoint representation for the su(2) subalgebra associated to the roots $\pm\alpha$.

   In pictures, this means that each pair of roots $\pm\alpha$ are "spokes" sticking out in opposite directions from a central hub that corresponds to the Cartan subalgebra. The 2D (rank $r = 2$) example of su(3) is shown in Fig. 4.1. Each su(2) subalgebra $\{e_\alpha, e_{-\alpha}, h_\alpha\}$, allows motion through the hub along the spokes $\pm\alpha$. What is missing from Fig. 4.1 are the allowed moves *between spokes corresponding to different roots.*

4.4.2 Construction of the Cartan subalgebra

The Cartan subalgebra is constructed by finding the **regular** element of a Lie algebra L. $h \in L$ is regular if $\widehat{ad}_h$ has as few zero eigenvalues as possible. In su(3), Eq. (3.2.7) implies that t_z is the regular element, since $\widehat{ad}_{t_z}$ has only two zero eigenvalues, while y has four.

The Cartan subalgebra H is then defined as the maximal abelian (commuting) subalgebra of L containing the regular element. In su(3), we add y [Eq. (3.2.7)] to get a rank-2 subalgebra.

4.4.3 Commutation relations

The above establishes Lie bracket relations between elements of H with themselves and with root vectors. What remains to be determined are brackets between two root vectors e_α and e_β. Assume $h \in H$ and consider

$$\begin{aligned} ad_h\left([e_\alpha, e_\beta]\right) &= [h, [e_\alpha, e_\beta]] \\ &= -[e_\alpha, [e_\beta, h]] - [e_\beta, [h, e_\alpha]] = [\alpha(h) + \beta(h)]\, [e_\alpha, e_\beta], \end{aligned} \tag{4.4.3}$$

where we have used the Jacobi identity [Eq. (1.2.1d)]. Eq. (4.4.3) applies for *any* h in the Cartan subalgebra. Then one of the following must be true:

$$\alpha + \beta = 0, \quad [e_\alpha, e_\beta] \neq 0, \qquad \Rightarrow \qquad [e_\alpha, e_{-\alpha}] \in H, \qquad \text{or} \tag{4.4.4a}$$

$$\alpha + \beta \neq 0, \quad [e_\alpha, e_\beta] = 0, \qquad \text{or} \tag{4.4.4b}$$

$$\alpha + \beta \neq 0, \quad [e_\alpha, e_\beta] \propto e_{\alpha+\beta}. \tag{4.4.4c}$$

The first case corresponds to $[s^+, s^-] = 2s^z$; $[e_\alpha, e_{-\alpha}] \in H$ can be regarded as the analog of s^z for this su(2) subalgebra. The second and third cases prove that a Lie bracket between two root vectors e_α and e_β either vanishes, or gives another root vector with root $\alpha + \beta$. If all brackets between root vectors with $\alpha + \beta \neq 0$ were to vanish, we would simply have a collection of independent su(2)s. In a simple Lie algebra of rank $r > 1$, there are nonvanishing brackets.

We can refine Eq. (4.4.4a) and make a connection to Eqs. (4.3.1) and (4.3.6), which involve root representation in H. First we prove two lemmas.

Lemma:

$$(e_\alpha, e_\beta) = 0 \;\text{ unless }\; \alpha + \beta = 0. \tag{4.4.5}$$

Proof: Evaluate the Killing form in the Cartan-Weyl basis, so that the trace is over root vectors and r Cartan subalgebra basis elements. For $h_i \in H$,

$$\begin{aligned} ad_{e_\alpha} ad_{e_\beta}(h_i) &= [e_\alpha, [e_\beta, h_i]] = -\beta(h_i)\, [e_\alpha, e_\beta] \\ &= \begin{cases} 0, & \text{if } [e_\alpha, e_\beta] = 0, \\ -\beta(h_i)\, \mathcal{C}_{\alpha,\beta}\, e_{\alpha+\beta}, & \text{if } [e_\alpha, e_\beta] \neq 0, \end{cases} \end{aligned} \tag{4.4.6}$$

where $\mathcal{C}_{\alpha,\beta}$ is a numerical constant. The only possible contribution to the trace (Killing form) must have $e_{\alpha+\beta} \in H$, i.e. $\alpha + \beta = 0$ [c.f. Eq. (4.2.2)]. For a root vector e_γ,

$$ad_{e_\alpha} ad_{e_\beta}(e_\gamma) = [e_\alpha, [e_\beta, e_\gamma]] = \mathcal{C}_{\beta,\gamma}[e_\alpha, e_{\beta+\gamma}] = \mathcal{C}_{\beta,\gamma}\, \mathcal{C}_{\alpha,\beta+\gamma}\, e_{\alpha+\beta+\gamma}. \tag{4.4.7}$$

Again, the only contribution to the trace must have $\alpha + \beta = 0$. Q.E.D.
Lemma: Invariance of the Killing form

$$(a, [b, c]) = ([a, b], c)\,. \tag{4.4.8}$$

Proof:

$$(a, [b, c]) = \mathsf{Tr}\left[\widehat{ad}_a\left[\widehat{ad}_b, \widehat{ad}_c\right]\right] = \mathsf{Tr}\left[\left[\widehat{ad}_a, \widehat{ad}_b\right]\widehat{ad}_c\right] = ([a, b], c)\,.$$

Now consider

$$\begin{aligned} ([e_\alpha, e_{-\alpha}], h) &= (e_\alpha, [e_{-\alpha}, h]) = \alpha(h)\,(e_\alpha, e_{-\alpha})\,, \\ \Rightarrow\ \alpha(h) &= \left(\frac{[e_\alpha, e_{-\alpha}]}{(e_\alpha, e_{-\alpha})}, h\right). \end{aligned} \tag{4.4.9}$$

Eq. (4.3.1) then implies that the root α is represented in the Cartan subalgebra by the element

$$\boxed{h_\alpha \equiv \frac{[e_\alpha, e_{-\alpha}]}{(e_\alpha, e_{-\alpha})}, \qquad \text{root } \alpha \text{ representation in the Cartan subalgebra } H.} \tag{4.4.10}$$

The number of positive roots is always larger than the rank r for $r > 1$, so that the set $\{h_\alpha\}$ forms an overcomplete basis for the Cartan subalgebra.

We apply Eq. (4.4.10) to su(3), relabeling $\{t_\pm, u_\pm, v_\pm\} \to \{e_{\pm\alpha_1}, e_{\pm\alpha_2}, e_{\pm\alpha_3}\}$. Eqs. (3.1.5) and (4.2.4) give

$$h_{\alpha_1} = \frac{t_z}{3}, \quad h_{\alpha_2} = \frac{3y - 2t_z}{12}, \quad h_{\alpha_3} = \frac{3y + 2t_z}{12},$$

identical to Eq. (4.3.5).

We can summarize the Lie brackets for all elements of a Lie algebra L in the Cartan-Weyl basis as follows. Let $\Sigma \equiv L - H$ denote the root vector space. Then we can write

$$[h_i, h_j] = 0, \qquad h_{i,j} \in H, \tag{4.4.11a}$$

$$[h_i, e_\alpha] = \alpha(h_i)\, e_\alpha, \qquad h_i \in H,\ \ e_\alpha \in \Sigma, \tag{4.4.11b}$$

$$[e_\alpha, e_\beta] = \begin{cases} (e_\alpha, e_{-\alpha})\, h_\alpha, & \alpha + \beta = 0, \\ N_{\alpha,\beta}\, e_{\alpha+\beta}, & \alpha + \beta \neq 0,\ e_{\alpha+\beta} \in \Sigma, \\ 0, & \alpha + \beta \neq 0,\ e_{\alpha+\beta} \notin \Sigma. \end{cases} \tag{4.4.11c}$$

The structure constants $\{N_{\alpha,\beta} \in \mathbb{C}\}$ are the nontrivial data that connect different su(2)s in L. In module 5 we will prove that $N_{\alpha,\beta} \neq 0$ if $\alpha + \beta$ is a root.

References

[1] Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Benjamin/Cummings, Menlo Park, California, 1984).