6. Cartan classification of Hamiltonians: the 10-fold way
* version 1.1 *

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Contents

This material is reviewed in (e.g.) [1, 2], which detail the classification of (strong, "symmetry-protected") topological insulators and superconductors in arbitrary spatial dimensions. These references clarified and expanded the connection between equivalence classes of bulk topological band structures on one hand, and robust surface state theories with anomalous local symmetries on the other. For a different approach that gives the same results, see [3]. Like bulk topological band structures, surface state theories also form equivalence classes, since these are immune to the most detrimental effects of quenched disorder (topological protection from Anderson localization) [1, 2, 4].

6.1 Random matrix theory and Cartan's classification: overview and modern applications

There are various situations in quantum physics where it makes sense to study an *ensemble of Hamiltonians*, instead of one particular Hamiltonian.[1] The simplest version applies to a system with a finite N-dimensional Hilbert space, such that the Hamiltonian $\hat{h}$ is an $N \times N$ Hermitian matrix. In **random matrix theory**, one studies ensemble-averaged properties of such matrices, such as the density of states and correlation functions between eigenvalues.

Random matrix theory [5, 6] was originally invented to study highly excited levels in complex nuclei. The idea was that many particles interact with each other in a confined volume, leading to an effectively random Hamiltonian constrained only by key

[1] Note this is different from the usual ergodic hypothesis for thermodynamics, where one averages over *states* for a fixed Hamiltonian.

symmetries. These should be symmetries unrelated to the Poincaré group of spacetime translations, boosts, and rotations, since all of these are broken if we consider the "effective single particle" Hamiltonian seen by a typical nucleon interacting with many others. Much later the same ideas were applied to irregularly shaped quantum dots formed in solids. A quantum dot is a meso- or nanoscopic "artificial atom" with an irregular boundary that can trap a single electron, such that the corresponding classical problem would exhibit conservative chaotic dynamics (point particle moving in a chaotic billiard).

Cartan showed that $N \times N$ random Hermitian matrices can be divided into 10 symmetry classes. Each class views a particular Hamiltonian $\hat{h}$ as either (1) an element of a Lie algebra (typically in the fundamental representation), or (2) an element of a Lie algebra coset representation. A coset is just a Lie algebra with some subalgebra removed. Different classes are realized by imposing different discrete symmetries that hold for every member of the class (such as time-reversal symmetry). These are imposed on the otherwise random form of $\hat{h}$. The 10 classes correspond to a special set of highly symmetric manifolds, the Riemannian symmetric spaces.

In fact, this "10-fold way" classification system has recently found much wider application in condensed matter physics. It can be used to classify "dirty" quantum solids, wherein electrons scatter randomly off of different types of *quenched disorder* such as impurities, vacancies, grain boundaries, etc. It turns out that many properties of such systems are independent of the details of disorder, and depend only on the spatial dimensionality d and the symmetry class [7].[2]

The 10-fold way is also useful for classifying topological band structures [1,2,4]. Here one considers a d-dimensional clean (= perfectly crystalline) non-interacting or mean-field model of electrons in a solid. If the system is fully gapped (i.e. $M \in \{1, 2, 3, \ldots\}$ bands are completely filled at zero temperature, with all others completely empty), then it turns out that the system can exhibit nontrivial topology with interesting physical implications. Such systems are called (strong) topological insulators and superconductors, and are predicted to host $(d-1)$-dimensional surface states with remarkable properties [4,8]. The latter circulate along the $(d-1)$-dimensional physical boundary of the bulk d-dimensional material sample. Why is the classification useful here? The answer is precisely because these systems are topological. It means that key properties such as the existence of surface states hold for an equivalence class of band structures, wherein you can move around the details of the energy bands in any fashion so long as you do not (1) close the energy gap or (2) break a defining symmetry.

In what follows, we will discuss only the classification scheme itself, seeing how it arises in concrete examples. We will not pursue calculations in random matrix theory, disordered and/or topological quantum phases. One of the most interesting directions is the connection between bulk topology and the physics of topological surface states in the presence of surface perturbations, such as quenched disorder and interactions. But that's another story...

6.2 Spinless fermions

6.2.1 Hamiltonian

Consider a tight-binding model of *spinless*, non-interacting fermions hopping on a lattice in d dimensions. This can be written generically through the second-quantized Hamiltonian

$$\hat{H} = \sum_{ij} \left[t_{ij} c_i^\dagger c_j + \frac{1}{2} \Delta_{ij} c_i^\dagger c_j^\dagger + \frac{1}{2} \Delta_{ij}^\dagger c_i c_j \right]. \tag{6.2.1}$$

Here c_i and $c_i^\dagger$ are fermion annihilation and creation operators satisfying canonical anticommutation relations

$$c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}. \tag{6.2.2}$$

Let us assume that the lattice has a finite number of N sites. Then $i, j \in \{1, 2, \ldots, N\}$. The $N \times N$ "hopping" matrix t_{ij} is Hermitian ($\hat{t}^\dagger = \hat{t}$), and encodes how electrons "tunnel" from site to site on the lattice. It can be translationally invariant in a clean system, or random in a dirty one. Typically we would let t_{ij} connect only sites that are within some fixed spatial radius, e.g. up to nearest-neighbor or next-nearest-neighbor hopping, but it is not necessary to specify this here. The anomalous terms proportional to $\hat{\Delta}$ and $\hat{\Delta}^\dagger$ arise in the mean-field description of superconductors, where Δ_{ij} is the superconducting gap parameter. It can also be translationally invariant or not. A normal metal or insulator has $\hat{\Delta} = \hat{\Delta}^\dagger = 0$.

It will prove useful to rewrite Eq. (6.2.1) in a more compact way. Let us introduce the "Majorana" spinor

$$\chi \equiv \begin{bmatrix} c \\ \left(c^\dagger\right)^\mathsf{T} \end{bmatrix} \quad \Rightarrow \quad \chi_{i,\sigma} = \begin{cases} c_i, & \sigma = 1 \\ c_i^\dagger, & \sigma = 2 \end{cases} \tag{6.2.3}$$

[2] We emphasize however that for an extended system in d spatial dimensions, most physical properties do not obtain from random matrix theory. The latter is useful for calculating ensemble-averaged observables in systems that are effectively "zero dimensional," such as a nucleus. Only the 10-fold classification scheme is useful for $d > 0$; physical properties need to be computed using other tools. For systems that are Anderson *delocalized*, these tools nevertheless always have a strongly Lie algebraic character, e.g. matrix non-linear sigma models [7], affine Lie current algebras, effective replica or graded supersymmetric Hamiltonians via transfer matrix, etc.

We collect all of the annihilation and creation operators and stack them in a single $2N$-component column vector; the result is χ. It carries two different types of indices: a site index $i \in \{1, 2, \ldots, N\}$, and a "particle-hole" index $\sigma \in \{1, 2\}$, such that $\sigma = 1$ ($\sigma = 2$) corresponds to an annihilation (creation) operator. In the first equality of Eq. (6.2.3), we have suppressed the site index. In this case, we view c ($c^\dagger$) as a column (row) vector in the ordered list of lattice sites. The symbol $(c^\dagger)^\mathsf{T}$ means to take the row vector $\begin{bmatrix} c_1^\dagger & c_2^\dagger & \cdots & c_N^\dagger \end{bmatrix}$ and transpose it to get a column, so that χ is a stack of two column vectors.

We introduce a set of block $2N \times 2N$ Pauli matrices acting on particle-hole space,

$$\hat{\sigma}_1 = \begin{bmatrix} 0 & \hat{1}_N \\ \hat{1}_N & 0 \end{bmatrix}, \quad \hat{\sigma}_2 = \begin{bmatrix} 0 & -i\hat{1}_N \\ i\hat{1}_N & 0 \end{bmatrix}, \quad \hat{\sigma}_3 = \begin{bmatrix} \hat{1}_N & 0 \\ 0 & -\hat{1}_N \end{bmatrix}. \tag{6.2.4}$$

Then, we note that

$$\chi^\dagger = \begin{bmatrix} c^\dagger & c^\mathsf{T} \end{bmatrix} = \chi^\mathsf{T} \hat{\sigma}_1. \tag{6.2.5}$$

Thus $\chi^\dagger$ is not independent of χ; for this reason we refer to χ as a "Majorana" fermion field.[3] Eq. (6.2.1) can then be rewritten as

$$H = \frac{1}{2} \chi^\mathsf{T} \hat{M}_P \, \hat{h} \, \chi, \qquad \hat{M}_P = \hat{\sigma}_1, \qquad \hat{h} = \begin{bmatrix} \hat{t} & \hat{\Delta} \\ \hat{\Delta}^\dagger & -\hat{t}^\mathsf{T} \end{bmatrix}. \tag{6.2.6}$$

The matrix $\hat{h} = \hat{h}^\dagger$ is the effective single-particle Hamiltonian; for a superconductor ($\hat{\Delta} \neq 0$), this is the **Bogoliubov-de Gennes** Hamiltonian.

6.2.2 Symmetries

The next step is to define symmetry conditions for $\hat{h}$. Since we wish to apply the classification scheme of random matrix theory, we exclude spatial symmetries like translations and rotations. Alternatively, we can imagine that these are explicitly broken by the presence of impurities or other forms of quenched disorder. What other symmetries can $\hat{h}$ possess? In fact, there is one symmetry that is "automatic:" **particle-hole** symmetry. Consider the following manipulation, wherein we anticommute the χ fields on the left and right-hand sides of H in Eq. (6.2.6):

$$\begin{aligned} H &= \frac{1}{2} \chi_i \left[\hat{M}_P\right]_{ij} \left[\hat{h}\right]_{jk} \chi_k \\ &= -\frac{1}{2} \chi_k \left[\hat{h}^\mathsf{T}\right]_{kj} \left[\hat{M}_P^\mathsf{T}\right]_{ji} \chi_i + \text{(const.)} \\ &= -\frac{1}{2} \chi^\mathsf{T} \hat{M}_P \hat{M}_P \, \hat{h}^\mathsf{T} \hat{M}_P^\mathsf{T} \chi \\ &= \frac{1}{2} \chi^\mathsf{T} \hat{M}_P \left(-\hat{M}_P \, \hat{h}^\mathsf{T} \, \hat{M}_P \right) \chi. \end{aligned} \tag{6.2.7}$$

Going from the first to second lines in this equation, we pick up an additive constant (const.) due to the non-trivial anticommutator Eq. (6.2.2); we drop this term in the following because it plays no role in the argument. To get the final equality, we have used the facts that $\hat{M}_P^2 = \hat{1}_{2N}$ and $\hat{M}_P^\mathsf{T} = \hat{M}_P$. We see that without loss of generality, we can take $\hat{h}$ to satisfy

$$-\hat{M}_P \, \hat{h}^\mathsf{T} \, \hat{M}_P = \hat{h}, \quad \hat{M}_P = \begin{cases} +\hat{M}_P^\mathsf{T}, & P^2 = +1 \\ -\hat{M}_P^\mathsf{T}, & P^2 = -1 \end{cases} \qquad \text{particle-hole symmetry } P;\ P^2 = \pm 1. \tag{6.2.8}$$

The notation $P^2 = +1$ ($P^2 = -1$) is used for the case of symmetric $\hat{M}_P$ (antisymmetric $\hat{M}_P$); these are unitarily inequivalent (cannot be interchanged by a basis transformation).[4] In our case, $\hat{M}_P = \hat{\sigma}_1 = \hat{M}_P^\mathsf{T}$ (symmetric). Physically, this "particle-hole" symmetry[5] appears because the anomalous terms in Eq. (6.2.1) satisfy (e.g.)

$$\frac{1}{2} \sum_{ij} \Delta_{ij} c_i^\dagger c_j^\dagger = -\frac{1}{2} \sum_{ij} \Delta_{ij} c_j^\dagger c_i^\dagger = \frac{1}{2} \sum_{ij} \left(-\Delta_{ji}\right) c_i^\dagger c_j^\dagger, \tag{6.2.9}$$

[3] The term "Majorana fermion" strictly speaking should apply only to a spinor field χ in a Lorentz covariant quantum field theory, wherein χ satisfies a condition as in Eq. (6.2.5). Here we are abusing high energy field theorist nomenclature to apply the terminology to a lattice model. In either case, these are just ordinary fermions satisfying a reality condition. There is something else that frequently gets called a "Majorana fermion," but is really a Majorana zero mode bound to a topological defect. These objects aren't fermions at all, since they satisfy non-abelian braiding statistics, and might be useful for topological quantum computation [9].

[4] The notational motivation for $P^2 = \pm 1$ and $T^2 = \pm 1$ obtains by considering the action of two successive particle-hole or time-reversal transformations on a fermion field operator in second quantization. The field must return to itself after two applications of the symmetry transformation, up to an overall sign. This sign determines the "square" of the symmetry operation.

[5] In high-energy physics, particle-hole symmetry is referred to as "charge conjugation" symmetry and denoted by C, in order to distinguish it from parity.

wherein we first anticommute the two creation operators, and then relabel the summation indices. As a result,

$$\hat{\Delta} = -\hat{\Delta}^{\mathsf{T}}, \tag{6.2.10}$$

i.e. the pairing gap must be an antisymmetric function of space. This is the well-known constraint that spinless fermions can only pair in odd angular momentum channels (p-wave,f-wave, etc.).

• **Exercise**: Verify that Eq. (6.2.8) implies Eq. (6.2.10). What happens if we take a more general Δ_{ij} that does not satisfy Eq. (6.2.10)?

Another important symmetry condition is **time-reversal invariance.** For our spinless lattice fermions in Eq. (6.2.1), time-reversal can be encoded as the transformation

$$c_i \rightarrow c_i, \quad c_j^\dagger \rightarrow c_j^\dagger, \quad i \rightarrow -i. \tag{6.2.11}$$

What this means is that the creation and annihilation operators are left unchanged,[6] but all complex-valued parameters in the model Eq. (6.2.1) are complex conjugated. The transformation law for χ [Eq. (6.2.3)] is also trivial,

$$\chi \rightarrow \chi, \qquad \chi^\dagger \rightarrow \chi^\dagger, \qquad i \rightarrow -i, \tag{6.2.12}$$

leading via Eq. (6.2.6) to

$$\hat{M}_T\,\hat{h}^*\,\hat{M}_T = \hat{h}, \quad \hat{M}_T = \begin{cases} +\hat{M}_T^{\mathsf{T}}, & T^2 = +1 \\ -\hat{M}_T^{\mathsf{T}}, & T^2 = -1 \end{cases} \qquad \text{time-reversal symmetry } T;\, T^2 = \pm 1. \tag{6.2.13}$$

In our case,

$$\hat{M}_T = \hat{1}_{2N} = \hat{M}_T^{\mathsf{T}}. \tag{6.2.14}$$

The antisymmetric case arises for spin-1/2 fermions, as we shall see below.

Finally, we can define a third possible symmetry, which is the product of time-reversal and particle-hole. Combining Eqs. (6.2.8) and (6.2.13) gives **chiral** (sometimes called "sublattice" [1, 2]) symmetry,

$$-\hat{M}_S\,\hat{h}\,\hat{M}_S = \hat{h}, \quad \hat{M}_S = \hat{M}_S^\dagger = \hat{M}_S^{-1} \qquad \text{chiral symmetry } S. \tag{6.2.15}$$

In our case

$$\hat{M}_S = \hat{\sigma}_1. \tag{6.2.16}$$

As suggested by Eqs. (6.2.8) and (6.2.13), for each of time-reversal and particle-hole symmetries, there are two inequivalent types. These are distinguished by the symmetry or antisymmetry of the conjugating matrices $\hat{M}_{P,T} = \pm\hat{M}_{P,T}$. The chiral symmetry S is simply present or not, and is always present if both P and T are.

The complete Cartan classification is formed by taking all possible combinations of T, P, and S: each of $\mathcal{O} \in \{T, P\}$ is absent, present with $\mathcal{O}^2 = +1$, or present with $\mathcal{O}^2 = -1$. That gives 9 possibilities. The final possibility has neither T nor P, but does possess chiral symmetry S. The classes are listed in Table 1, along with example theoretical and or experimental realizations in two- and three-dimensional condensed matter systems.

• A word of caution: Although the set of $\{T, P, S\}$ exhausts the possibilities, it is not always true that T and P correspond to the *physical* symmetries for a given model. In particular, we will see that if there is an additional continuous symmetry present, it can change the character of the *effective form* of T or P used in the classification Table 1. In fact, if there is a continuous symmetry, one should first block diagonalize $\hat{h}$ using that symmetry, and then reconsider the classification *for the irreducible blocks*. Nevertheless, we can realize all possible classes by considering spinless and spin-1/2 Hamiltonians with different physical symmetries.

6.2.3 Examples

6.2.3.1 Spinless superconductor without time-reversal invariance: Class *D*

Consider a system with Hamiltonian H [Eq. (6.2.1)] and no physical symmetries. You might expect this to correspond to a metal with broken time-reversal symmetry ("unitary class", see below), but this is not the case. Recall that any H cast in the form of

[6] Note that Eq. (6.2.11) encodes the transformation in second quantization on the operators and parameters. It is *not* a transformation on wavefunctions. A wavefunction ψ has to be complex-conjugated under a time-reversal transformation, but an annihilation operator c_i does not become a creation operator. There is a symmetry operation that exchanges creation and annihilation operations; this is in fact the second-quantized version of particle-hole symmetry [Eq. (6.2.8)].

the Bogoliubov-de Gennes equation in Eq. (6.2.6) is automatically invariant under P, Eq. (6.2.8) with $\hat{M}_P = \hat{\sigma}_1$. Physically, a superconductor has *less symmetry* than a metal, because it lacks U(1) electric charge symmetry; the latter is broken spontaneously. If there are no other symmetries, then we can only impose Eq. (6.2.8); explicitly,

$$-\hat{\sigma}_1\,\hat{h}^{\mathsf{T}}\,\hat{\sigma}_1 = \hat{h}, \qquad P^2 = 1 \text{ particle-hole symmetry.} \tag{6.2.17}$$

Now, $\hat{h}$ is a $2N \times 2N$ Hermitian matrix. It can therefore be expressed as a linear combination of Hermitian generators for the group U($2N$), acting on vectors in the defining $2N$ representation. We write this as

$$\hat{h} \in \mathrm{u}(2N),$$

since $\hat{h}$ is an element of the Lie algebra. Using Eqs. (2A.2.2) and (2A.2.3) from module 2A, Eq. (6.2.17) implies that

$$\hat{h} \in \mathrm{so}(2N), \qquad \text{spinless superconductor with } P^2 = +1, \text{ broken time-reversal: class } D. \tag{6.2.18}$$

In other words, $\hat{h}$ is formed from a subset of u($2N$) generators, and these are precisely the generators of the subgroup so($2N$). In the Cartan classification, this is an example of class D, since D_n corresponds to the Lie algebra so($2N$) [Eq. (5.2.12)].

6.2.3.2 Spinless time-reversal invariant superconductor: Class *BDI*

We can also consider a time-reversal invariant spinless superconductor. Then we must also enforce Eq. (6.2.13),

$$\hat{h}^{\mathsf{T}} = \hat{h}, \qquad T^2 = 1 \text{ time-reversal symmetry.} \tag{6.2.19}$$

Consider the *complementary* condition

$$-\hat{h}^{\mathsf{T}} = \hat{h}. \tag{6.2.20}$$

If we were to enforce particle-hole [Eq. (6.2.17)] and the "anti-time-reversal symmetry" in Eq. (6.2.20), then we would have

$$-\hat{h}^{\mathsf{T}} = \hat{h}, \tag{6.2.21a}$$

$$\hat{\sigma}_1\,\hat{h}\,\hat{\sigma}_1 = \hat{h}. \tag{6.2.21b}$$

We can make a change of basis using $\hat{h}' \equiv \hat{U}^\dagger\,\hat{h}\,\hat{U}$, where $\hat{U} = (1/\sqrt{2})(\hat{1}_{2N} - i\hat{\sigma}_2)$, leading to

$$-\hat{h}'^{\mathsf{T}} = \hat{h}', \tag{6.2.22a}$$

$$\hat{\sigma}_3\,\hat{h}'\,\hat{\sigma}_3 = \hat{h}'. \tag{6.2.22b}$$

• **Exercise**: Verify Eq. (6.2.22).
Eq. (6.2.22) implies that we can make the σ-space decomposition

$$\hat{h}' = \begin{bmatrix} \hat{h}_N^{(1)} & 0 \\ 0 & \hat{h}_N^{(2)} \end{bmatrix}, \quad \left[\hat{h}_N^{(i)}\right]^{\mathsf{T}} = -\hat{h}_N^{(i)} \;\; (i \in \{1,2\}). \tag{6.2.23}$$

Since the $N \times N$ matrices $\hat{h}_N^{(1)}$ and $\hat{h}_N^{(2)}$ appear in separate blocks of $\hat{h}'$, they represent independent, mutually commuting algebras. Moreover, each satisfies an orthogonal condition. Enforcing particle-hole [Eq. (6.2.17)] and "anti-time-reversal symmetry" [Eq. (6.2.20)] would therefore give $\hat{h} \in \mathrm{so}(N) \times \mathrm{so}(N)$. But this isn't what we want: we want to enforce time-reveral symmetry Eq. (6.2.19). We therefore conclude that

$$\hat{h} \in \mathrm{so}(2N)\,/\,\mathrm{so}(N) \times \mathrm{so}(N), \qquad \text{spinless superconductor with } T^2 = +1,\ P^2 = +1\text{: class } BDI. \tag{6.2.24}$$

Here the quotient notation g/h for Lie algebras g and $h \in g$ means the subset (coset) of g generators that are <u>not</u> in the subalgebra h. The Cartan label for this class is *BDI*. The corresponding manifold SO($2N$)/SO(N) $\times$ SO(N) is also known as the orthogonal Grassmannian.

6.2.3.3 Metal with broken time-reversal symmetry: "Unitary" class A

Next we consider a metal, Eqs. (6.2.1) and (6.2.6) with $\hat{\Delta} = 0$. In this case $\hat{h}$ is block diagonal in σ-space. Equivalently, we can say that $\hat{h}$ commutes with the generator of U(1) electric charge rotation. An electric charge rotation is

$$c_i \rightarrow \exp(i\theta)\, c_i, \quad c_i^\dagger \rightarrow \exp(-i\theta)\, c_i^\dagger, \qquad \Rightarrow \qquad \chi \rightarrow \exp\left(i\theta\hat{\sigma}_3\right)\, \chi, \tag{6.2.25}$$

so that invariance of Eq. (6.2.6) requires that

$$\hat{\sigma}_3\, \hat{h}\, \hat{\sigma}_3 = \hat{h}, \qquad \text{electric U(1) invariance.} \tag{6.2.26}$$

Formally, we can determine the structure of $\hat{h}$ by combining the built-in particle-hole condition Eq. (6.2.17) with electric U(1) invariance Eq. (6.2.26). The latter implies the block decomposition

$$\hat{h} = \begin{bmatrix} \hat{h}_N^{(1)} & 0 \\ 0 & \hat{h}_N^{(2)} \end{bmatrix}. \tag{6.2.27}$$

By itself, this would imply $\hat{h} \in \mathrm{u}(N) \times \mathrm{u}(N)$. Imposing Eq. (6.2.17) constrains

$$-\left[\hat{h}_N^{(2)}\right]^{\mathsf{T}} = \hat{h}_N^{(1)}, \tag{6.2.28}$$

which implies that the blocks are not independent. Since there is no further constraint on $\hat{h}_N^{(1)}$, we conclude that

$$\boxed{\hat{h} \in \mathrm{u}(N), \qquad \text{metal with broken } T\text{: class } A.} \tag{6.2.29}$$

Again the Cartan notation comes from the classification of Lie algebras, since $A_n = \mathrm{su}(n+1)$.

Here we have an example of the warning given at the end of Sec. 6.2.2: the combination of the discrete P symmetry with continuous U(1) symmetry effectively gives a subblock Hamiltonian with *no* T, P, or S symmetry. Class A is also called the "unitary" class, and is one of the original three random matrix classes investigated a half-century ago by Wigner and Dyson. (The others are the "orthogonal" AI and "symplectic" AII classes, described below; see also Table 1). The additional 7 classes in Table 1 were only systematically introduced into condensed matter physics in the 90s by Altland and Zirnbauer [7].

6.2.3.4 Spinless metal with time-reversal symmetry: "Orthogonal" class AI

Finally, we consider the combination of P, T, and electric U(1) symmetry. As shown above, P and electric U(1) give $\hat{h} \in u(N)$. The "anti-time-reversal" invariance in Eq. (6.2.20) would give $\hat{h} \in \mathrm{so}(N)$. Therefore

$$\boxed{\hat{h} \in \mathrm{u}(N)\,/\,\mathrm{so}(N), \qquad \text{spinless metal with } T^2 = +1\text{: class } AI.} \tag{6.2.30}$$

This is the "orthogonal" Wigner-Dyson class, c.f. Table 1.

6.3 Spin-1/2 fermions

6.3.1 Hamiltonian

We can also consider spin-1/2 fermions. The Bogoliubov-de Gennes Hamiltonian is

$$\hat{H} = \sum_{ij} \left[t_{is\,js'} c_{i,s}^\dagger c_{j,s'} + \frac{1}{2}\Delta_{is\,js'} c_{i,s}^\dagger c_{j,s'}^\dagger + \frac{1}{2}\Delta_{is\,js'}^\dagger c_{i,s} c_{j,s'} \right]. \tag{6.3.1}$$

Here $s, s' \in \{\uparrow, \downarrow\}$ denote the spin-1/2 state along the z-axis in spin space. The hopping $\hat{t}$ and pairing $\hat{\Delta}$ amplitudes are now matrices in both position and spin space. Spin-dependent hopping arises with spin-orbit coupling, while spin-triplet pairing can break spin SU(2) symmetry in $\Delta_{is\,js'}$. Again we re-express this using a "Majorana" field that collects all degrees of freedom,

$$\chi \equiv \begin{bmatrix} c \\ i\hat{s}_2 \left(c^\dagger\right)^{\mathsf{T}} \end{bmatrix} \quad \Rightarrow \quad \chi_{i,\sigma,s} = \begin{cases} c_{i,s}, & \sigma = 1 \\ i\left[\hat{s}_2\right]_{s,s'} c_{i,s'}^\dagger, & \sigma = 2 \end{cases} \tag{6.3.2}$$

Here $\hat{s}_2$ is the antisymmetric Pauli matrix acting in spin space. Recall that $i\hat{s}_2$ serves as a "metric" for the $s = 1/2$ representation of SU(2), see Sec. 1.5 in module 1. It means that under a spin SU(2) transformation, χ defined in Eq. (6.3.2) transforms like c:

$$c \to \exp\left(-i\frac{\hat{\boldsymbol{s}}}{2}\cdot\boldsymbol{\theta}\right) c \quad \Rightarrow \quad \chi \to \exp\left(-i\frac{\hat{\boldsymbol{s}}}{2}\cdot\boldsymbol{\theta}\right)\chi. \tag{6.3.3}$$

• **Exercise**: Verify the χ-field spin SU(2) transformation in Eq. (6.3.3).
We note that

$$\chi^\dagger = \begin{bmatrix} c^\dagger & -ic^\mathsf{T}\hat{s}_2 \end{bmatrix} = \begin{bmatrix} c^\mathsf{T} & -ic^\dagger\hat{s}_2 \end{bmatrix}\hat{s}_2\hat{\sigma}_2 = \chi^\mathsf{T}\hat{s}_2\hat{\sigma}_2. \tag{6.3.4}$$

The Hamiltonian can be written as

$$H = \frac{1}{2}\chi^\mathsf{T}\hat{M}_P\,\hat{h}\,\chi, \qquad \hat{M}_P = \hat{\sigma}_2\hat{s}_2, \qquad \hat{h} = \begin{bmatrix} \hat{t} & -i\,\hat{\Delta}\,\hat{s}_2 \\ i\,\hat{s}_2\,\hat{\Delta}^\dagger & -\hat{s}_2\,\hat{t}^\mathsf{T}\,\hat{s}_2 \end{bmatrix}. \tag{6.3.5}$$

The key symmetry conditions are

1. P in Eq. (6.2.8) with

$$\hat{M}_P = \hat{\sigma}_2\hat{s}_2 = \hat{M}_P^\mathsf{T}, \qquad P^2 = +1 \text{ particle-hole "symmetry."} \tag{6.3.6}$$

2. T in Eq. (6.2.13) with

$$\hat{M}_T = \hat{s}_2, \qquad T^2 = -1 \text{ time-reversal symmetry.} \tag{6.3.7}$$

This arises due to the Kramers degeneracy of states related by time-reversal in a spin-1/2 system. Time-reversal is encoded in the transformation

$$c \to i\hat{s_2}c, \qquad c^\dagger \to c^\dagger(-i)\hat{s}_2, \qquad i \to -i. \tag{6.3.8}$$

We can also define a chiral condition from the product of P and T, but we don't need this in the examples that follow.

6.3.2 Examples

6.3.2.1 Spin-1/2 superconductor with no physical symmetries: Class *D*

We again consider a superconductor with no physical symmetries. Apriori,

$$\hat{h} \in \mathrm{u}(4N),$$

where $4N$ arises from the direct product of N lattice, 2 particle-hole, and 2 spin degrees of freedom. Imposing the automatic particle-hole condition in Eqs. (6.2.8) and (6.3.6) gives

$$\hat{h} \in \mathrm{so}(4N), \qquad \text{spin-1/2 superconductor with } P^2 = +1\text{, broken time-reversal, and spin-orbit coupling: class } D. \tag{6.3.9}$$

This is the same class as the spinless case without physical symmetries, Eq. (6.2.18).

6.3.2.2 Spin-1/2 superconductor with time-reversal symmetry and spin-orbit coupling: Class *DIII*

If we add time-reversal symmetry to the class D superconductor, we get the additional constraint implied by Eqs. (6.2.13) and (6.3.7). The "anti-time-reversal" condition would be

$$-\hat{s}_2\,\hat{h}^\mathsf{T}\,\hat{s}_2 = \hat{h}. \tag{6.3.10}$$

Combining this with particle-hole [Eqs. (6.2.8) and (6.3.6)] would give the additional constraint

$$\hat{\sigma}_2\,\hat{h}\,\hat{\sigma}_2 = \hat{h}. \tag{6.3.11}$$

Using a basis transformation $\hat{h}' \equiv \hat{U}^\dagger\,\hat{h}\,\hat{U}$, $\hat{U} = (1/\sqrt{2})(\hat{1}_{4N} + i\hat{\sigma}_1)$, Eqs. (6.3.10) and (6.3.11) become

$$-\hat{s}_2\hat{\sigma}_1\,\hat{h}'^\mathsf{T}\,\hat{s}_2\hat{\sigma}_1 = \hat{h}', \tag{6.3.12a}$$
$$\hat{\sigma}_3\,\hat{h}'\,\hat{\sigma}_3 = \hat{h}'. \tag{6.3.12b}$$

These imply the σ-space decomposition

$$\hat{h}' = \begin{bmatrix} \hat{h}^{(1)}_{2N} & 0 \\ 0 & \hat{h}^{(2)}_{2N} \end{bmatrix}, \qquad -\hat{s}_2 \left[\hat{h}^{(2)}_{2N}\right]^{\mathsf{T}} \hat{s}_2 = \hat{h}^{(1)}_{2N}. \tag{6.3.13}$$

The latter condition implies $\hat{h}^{(2)}_{2N}$ is not independent. Since there are no constraints on $\hat{h}^{(1)}_{2N}$, we would conclude $\hat{h} \in \mathrm{u}(2N)$. For the opposite case of time-reversal and particle-hole, we get

$$\hat{h} \in \mathrm{so}(4N)\,/\,\mathrm{u}(2N), \qquad \text{spin-1/2 superconductor with } T^2 = -1,\ P^2 = +1, \text{ and spin-orbit coupling: class } \mathit{DIII}. \tag{6.3.14}$$

Thus fermion quasiparticles in a spin-1/2 superconductor with time-reversal invariance and spin-orbit coupling [no spin SU(2) symmetry] belong to class *DIII*. Class *DIII* is realized by the 3D topological superfluid ^{3}He B, see Table 1 and Refs. [1,2,4,10]. Note that the matrix structure of this class is different from that of the spinless, time-reversal invariant superconductor (Sec. 6.2.3.2), owing to the $T^2 = -1$ version of time-reversal in the case of *DIII*.

6.3.2.3 Spin-1/2 metal with time-reversal invariance and spin-orbit coupling: "Symplectic" class *AII*

Next, we consider a metal with time-reversal symmetry and no spin SU(2) symmetry. We can view this as a restriction of class *DIII*, where we impose charge U(1) invariance:

$$\hat{\sigma}_3\, \hat{h}\, \hat{\sigma}_3 = \hat{h}. \tag{6.3.15}$$

Adding P and "anti-time-reversal" conditions gives Eqs. (6.3.10) and (6.3.11). The combination of Eqs. (6.3.15) and (6.3.11) imply that $\hat{h}$ is an identity matrix in σ-space, i.e. $\hat{h} \in \mathrm{u}(2N)$. Adding Eq. (6.3.10) would restrict

$$\hat{h} \in \mathrm{sp}(2N), \tag{6.3.16}$$

where we have used Eq. (2A.3.4) in module 2A. The combination of electric U(1) [Eq. (6.3.15)] and P [Eqs. (6.2.8) and (6.3.6)] alone implies

$$\hat{h} = \begin{bmatrix} \hat{h}^{(1)}_{2N} & 0 \\ 0 & \hat{h}^{(2)}_{2N} \end{bmatrix}, \qquad -\hat{s}_2 \left[\hat{h}^{(2)}_{2N}\right]^{\mathsf{T}} \hat{s}_2 = \hat{h}^{(1)}_{2N}, \tag{6.3.17}$$

so that $\hat{h} \in \mathrm{u}(2N)$. Quotienting out Eq. (6.3.16) finally gives

$$\hat{h} \in \mathrm{u}(2N)\,/\,\mathrm{sp}(2N), \qquad \text{spin-1/2 metal with } T^2 = -1 \text{ and spin-orbit coupling: class } \mathit{AII}. \tag{6.3.18}$$

This is the "symplectic" Wigner-Dyson class, c.f. Table 1. The recently-discovered $\mathbb{Z}_2$ topological insulators [8] in 2D and 3D reside in class *AII*.

6.3.2.4 Spin-1/2 metal with time-reversal invariance and spin SU(2) symmetry: "Orthogonal" class *AI*

Finally, we consider a metal with electric U(1) invariance, time-reversal symmetry, and spin SU(2) symmetry. The physical symmetries are

$$\hat{s}_2\, \hat{h}^{\mathsf{T}}\, \hat{s}_2 = \hat{h}, \qquad T^2 = -1 \text{ time-reversal invariance}, \tag{6.3.19a}$$

$$\hat{\sigma}_3\, \hat{h}\, \hat{\sigma}_3 = \hat{h}, \qquad \text{electric U(1) invariance}, \tag{6.3.19b}$$

$$-\hat{s}_2\hat{\sigma}_2\, \hat{h}^{\mathsf{T}}\, \hat{s}_2\hat{\sigma}_2 = \hat{h}, \qquad P^2 = +1 \text{ particle-hole (automatic)}, \tag{6.3.19c}$$

$$\hat{s}_i\, \hat{h}\, \hat{s}_i = \hat{h}, \quad i \in \{1,2,3\}, \qquad \text{spin SU(2) invariance}. \tag{6.3.19d}$$

Apriori $\hat{h} \in \mathrm{u}(4N)$, but the last spin SU(2) condition implies it must be proportional to the identity in spin space. We can then replace the above conditions with

$$\hat{h}^{\mathsf{T}} = \hat{h}, \qquad \underline{\textit{effective}}\ T^2 = +1 \text{ time-reversal invariance}, \tag{6.3.20a}$$

$$\hat{\sigma}_3\, \hat{h}\, \hat{\sigma}_3 = \hat{h}, \qquad \text{electric U(1) invariance}, \tag{6.3.20b}$$

$$-\hat{\sigma}_2\, \hat{h}^{\mathsf{T}}\, \hat{\sigma}_2 = \hat{h}, \qquad \underline{\textit{effective}}\ P^2 = -1 \text{ particle-hole (automatic)}. \tag{6.3.20c}$$

Here we see an example wherein the physical T and P are replaced with different *effective* T and P, after imposing spin SU(2) symmetry. In fact the P symmetry can be eliminated by further exploiting the electric U(1) invariance; these conditions imply that

$$\hat{h} = \begin{bmatrix} \hat{h}_N^{(1)} & 0 \\ 0 & \hat{h}_N^{(2)} \end{bmatrix}, \qquad -\left[\hat{h}_N^{(2)}\right]^{\mathsf{T}} = \hat{h}_N^{(1)}, \tag{6.3.21}$$

leading to $\hat{h} \in \mathrm{u}(N)$. Finally, we remove the subalgebra associated to the "anti-time-reversal" condition $-\hat{h}^{\mathsf{T}} = \hat{h}$, leading to

$$\hat{h} \in \mathrm{u}(N)\,/\,\mathrm{so}(N), \qquad \text{spin-1/2 metal with } T^2 = -1 \text{ and spin SU(2) symmetry: class } AI. \tag{6.3.22}$$

In this case, the spinless (Sec. 6.2.3.4) and spinful classes for time-reversal invariant metals are identical, so long as spin SU(2) symmetry is imposed in the latter case. This is *not* true for superconductors, however. The generic class for spinless time-reversal invariant superconductors is class *BDI* [Eq. (6.2.24)]. A spin-1/2 time-reversal invariant superconductor with spin SU(2) symmetry instead resides in class *CI*, see Table 1.

• **Exercise**: Demonstrate that a spin SU(2) invariant superconductor without time-reversal symmetry (neglecting Zeeman effects) has

$$\hat{h} \in \mathrm{sp}(2N), \tag{6.3.23}$$

while a spin SU(2) invariant superconductor with physical time-reversal symmetry [Eqs. (6.2.13) and (6.3.7)] has

$$\hat{h} \in \mathrm{sp}(2N)\,/\,\mathrm{u}(N). \tag{6.3.24}$$

Verify that these correspond to classes C and CI in Table 1, with the prescribed *effective* versions of T and/or P.

6.4 The Cartan classification

Table 1: The "10-fold way:" the random matrix classification of Hamiltonians. The Cartan designation for the Riemannian symmetric space AI, DIII, etc. is given in the first column. Example realizations from condensed matter physics are listed in the second. These consist of two- or three-dimensional phases in non-interacting or mean field fermion models, or real (interacting) experimental systems in the same class. Here "SC" abbreviates superconductor. Real experimental systems are highlighted in blue. The defining discrete symmetries are listed next: T (time-reversal), P (particle-hole), and S (chiral or "sublattice"). As emphasized in the text, these are not necessarily the microscopic symmetries, if the system has an additional continuous symmetry. The matrix structure is listed in the penultimate column. The final column asks whether a gapped, strong topological phase (insulator or superconductor) can be realized in this class in three spatial dimensions.

Class	Example	T	P	S	Structure	Strong top. phase in 3D?
A "Unitary"	Metal with broken T due to (e.g.) magnetic impurities, or magnetic field: 2D integer quantum Hall effect	0	0	0	u(N)	no
D	SC with broken T; 2D topological p+ip SC: 5/2 state in fractional quantum Hall effect (?); superfluid ^{3}He *A*; 2D thermal quantum Hall effect	0	1	0	so($2N$)	no
C	SC with broken T and spin su(2) symmetry; 2D spin quantum Hall effect	0	−1	0	sp($2N$)	no
AI "Orthogonal"	Metal with T and spin su(2) symmetry: 3D P-doped Si, 2D Si MOSFET MIT	1	0	0	u(N) / so(N)	no
BDI	Orthogonal metal with chiral symmetry; spinless p-wave SC with T	1	1	1	so($2N$) / so(N)×so(N)	no
CI	SC with T and spin su(2) symmetry: 2D high-T_c cuprate SCs, 3D topological SC	1	−1	1	sp($2N$) / u(N)	yes
AII "Symplectic"	Metal with T and spin-orbit coupling; $\mathbb{Z}_2$ topological insulators in 2D, 3D	−1	0	0	u($2N$) / sp($2N$)	yes
DIII	SC with T and spin-orbit coupling: 3D topological SC, superfluid ^{3}He *B*	−1	1	1	so($2N$) / u(N)	yes
CII	Symplectic metal with chiral symmetry	−1	−1	1	sp($4N$) / sp($2N$)×sp($2N$)	yes
AIII	Unitary metal with chiral symmetry; SC with T and spin U(1) symmetry (e.g. p-wave); 3D topological SC	0	0	1	u($2N$) / u(N)×u(N)	yes

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