7. The classical and exceptional Lie algebras
* version 1.4 *

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In Sec. 5.3, we demonstrated how the root system (adjoint representation) for a generic Lie algebra can be constructed from the Cartan matrix or Dynkin diagram. The latter encodes the inner products and norm ratios of the simple roots. In this module, we assemble the Cartan matrix from the simple roots for the four classical Lie algebra families A_n, B_n, C_n, and D_n. The needed data is extracted from the defining representation for each algebra. We will also state (but not derive) the results for the five exceptional algebras G_2, F_4, and $E_{6,7,8}$.

In the next module 8, we will show how the Cartan matrix can be used to obtain the full set of weights in any (finite dimensional) irreducible representation. In addition, the weights of the defining representation for the orthogonal algebras B_n [so($2n+1$)] and D_n [so($2n$)] turn out to give a useful alternative basis for H^*, since these prove to be orthonormal. We will exploit this fact later in module 10 to construct spinor representations.

The discussion here largely follows chapter VIII of [1].

7.1 su(n): A_{n-1}

Consider the space of all $n \times n$ matrices with real or complex elements. A basis for this space can be defined as follows:

$$\left(\hat{E}_{ab}\right)_{ij} = \delta_{ai}\delta_{bj} \quad \Rightarrow \quad \hat{E}_{ab} \rightarrow \begin{array}{c} \\ \vdots \\ a-1 \\ a \\ a+1 \\ \vdots \\ \\ \end{array} \overset{\begin{array}{ccccc} \cdots & b-1 & b & b+1 & \cdots \end{array}}{\begin{bmatrix} & & & & \\ & & & & \\ & & 1 & & \\ & & & & \\ & & & & \end{bmatrix}}. \tag{7.1.1}$$

All unmarked entries are zero. These elements satisfy the multiplication rule

$$\hat{E}_{ab} \cdot \hat{E}_{cd} \to \left(\hat{E}_{ab}\right)_{ij} \left(\hat{E}_{cd}\right)_{jk} = \delta_{ai}\, \delta_{bj}\, \delta_{cj}\, \delta_{dk} \to \delta_{bc}\, \hat{E}_{ad}, \tag{7.1.2}$$

so that

$$[\hat{E}_{ab}, \hat{E}_{cd}] = \delta_{bc}\, \hat{E}_{ad} - \delta_{da}\, \hat{E}_{cb}. \tag{7.1.3}$$

Note that

$$\sum_c [\hat{E}_{ab}, \hat{E}_{cc}] = 0, \tag{7.1.4}$$

which implies that the identity

$$\hat{1}_n = \sum_c \hat{E}_{cc} \tag{7.1.5}$$

generates a one-dimensional abelian ideal. We want to construct root systems for generic semi-simple Lie algebras (no abelian ideals); we should therefore restrict ourselves to traceless $n \times n$ matrices, i.e. the su(n)= A_{n-1} Lie algebra. In this case, an arbitrary element of the Cartan subalgebra can be written as

$$\hat{H} = \sum_{i=1}^{n} \lambda_i\, \hat{E}_{ii}, \qquad \sum_{i=1}^{n} \lambda_i = 0. \tag{7.1.6}$$

Now

$$\widehat{ad}_h\,(e_{ab}) \to [\hat{H}, \hat{E}_{ab}] = \lambda_a\, \hat{E}_{ab} - \lambda_b\, \hat{E}_{ab}, \tag{7.1.7}$$

which implies that e_{ab} is a root vector:

$$[h, e_{ab}] = \alpha_{ab}(h)\, e_{ab}, \qquad \alpha_{ab}\left(\sum_i \lambda_i e_{ii}\right) = \lambda_a - \lambda_b. \tag{7.1.8}$$

We identify the positive root vectors with set of upper triangular $\{\hat{E}_{ab}\}$. For a generic Cartan subalgebra element $h \in H$ defined via Eq. (7.1.6), we claim that the set of simple roots Π can be associated to the first diagonal root vectors:

root vector	root $\alpha(h)$
$e_{12} \equiv e_{\overline{\alpha}_1}$	$\overline{\alpha}_1(h) = \lambda_1 - \lambda_2$
$e_{23} \equiv e_{\overline{\alpha}_2}$	$\overline{\alpha}_2(h) = \lambda_2 - \lambda_3$
$\vdots$	$\vdots$
$e_{n-1,n} \equiv e_{\overline{\alpha}_{n-1}}$	$\overline{\alpha}_{n-1}(h) = \lambda_{n-1} - \lambda_n$

(7.1.9)

Proof: A generic positive root is

$$\alpha_{ij}(h) = \lambda_i - \lambda_j, \quad i < j. \tag{7.1.10}$$

This can be written as

$$\alpha_{ij}(h) = (\lambda_i - \lambda_{i+1}) + (\lambda_{i+1} - \lambda_{i+2}) + (\lambda_{i+2} - \lambda_{i+3}) + \ldots + (\lambda_{j-1} - \lambda_j) = \sum_{l=i}^{j-1} \overline{\alpha}_l. \tag{7.1.11}$$

Therefore every positive root can be expressed as a linear combination of the $\{\overline{\alpha}_i\}$ with non-negative integral coefficients. Moreover, $\overline{\alpha}_i$ cannot be expressed as a linear combination of the other proposed simple roots. To see this, assume the converse, i.e. that $\overline{\alpha}_i$ is positive, but not simple. Then

$$\overline{\alpha}_i(h) = \lambda_i - \lambda_{i+1} = \sum_{j \neq i} \kappa_j^{(i)} \overline{\alpha}_j(h) = \ldots + \kappa_{i-1}^{(i)}(\lambda_{i-1} - \lambda_i) + \kappa_{i+1}^{(i)}(\lambda_{i+1} - \lambda_{i+2}) + \ldots \tag{7.1.12}$$

Since all the $\kappa_j^{(i)}$ are positive, there is no way to get λ_{i+1} with a negative coefficient (contradiction).

For the simple roots, we therefore have the associated Lie brackets

$$[e_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] = \delta_{j,i+1}\, e_{i,i+2} - \delta_{j,i-1}\, e_{i-1,i+1}, \tag{7.1.13a}$$
$$[e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_j}] = \delta_{ij}\,(e_{ii} - e_{i+1,i+1}), \tag{7.1.13b}$$
$$[h, e_{\overline{\alpha}_i}] = (\lambda_i - \lambda_{i+1})\, e_{\overline{\alpha}_i}, \quad h \text{ defined as in Eq. (7.1.6).} \tag{7.1.13c}$$

Here the root vector associated to $-\overline{\alpha}_j$ is $e_{-\overline{\alpha}_j} \to \hat{E}_{j+1,j}$. Eq. (7.1.13b) is consistent with the fact that simple roots are lowest weight states in mutually generated strings of the adjoint representation [Eq. (5.3.3)].

It is useful to construct the $\{h_{\overline{\alpha}_i}\}$ ($i \in \{1, 2, \ldots, n-1\}$), i.e. the elements of the Cartan subalgebra H that represent the simple roots. Consider the Killing form between generic elements of H. We use the algorithm described in module 4 [Eq. (4.2.2)]. For

$$h \equiv \sum_i \lambda_i e_{ii}, \quad h' \equiv \sum_j \lambda'_j e_{jj}, \tag{7.1.14}$$

we have the Killing form

$$(h, h') = \sum_{\substack{a,b=1 \\ a\neq b}}^{n} \Big([h, [h', e_{ab}]] \Big)_{\text{coeff. of } e_{ab}} = \sum_{a,b=1}^{n} (\lambda_a - \lambda_b)(\lambda'_a - \lambda'_b) = 2n \sum_{a=1}^{n} \lambda_a \lambda'_a, \tag{7.1.15}$$

since $\sum_b \lambda_b = 0$. Therefore, since

$$(h_{\overline{\alpha}_i}, h') = \overline{\alpha}_i(h') = \lambda'_i - \lambda'_{i+1}, \tag{7.1.16}$$

we conclude that

$$h_{\overline{\alpha}_i} = \frac{e_{ii} - e_{i+1,i+1}}{2n} \equiv \sum_{j=1}^{n} \lambda_j^{(\overline{\alpha}_i)} e_{jj}, \quad \lambda_j^{(\overline{\alpha}_i)} = \frac{1}{2n}(\delta_{i,j} - \delta_{i+1,j}). \tag{7.1.17}$$

Recall that [Eq. (4.4.10)]

$$h_{\overline{\alpha}_i} = \frac{[e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_i}]}{(e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_i})}. \tag{7.1.18}$$

Eqs. (7.1.13b), (7.1.17) and (7.1.18) therefore give the Killing form of the simple root vectors,

$$(e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_i}) = 2n. \tag{7.1.19}$$

We can now compute the scalar products between simple roots, using

$$\begin{aligned} [h_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] &= \langle \overline{\alpha}_i, \overline{\alpha}_j \rangle e_{\overline{\alpha}_j} \\ &= \left(\lambda_j^{(\overline{\alpha}_i)} - \lambda_{j+1}^{(\overline{\alpha}_i)}\right) e_{\overline{\alpha}_j} = \frac{1}{2n}(\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1} + \delta_{i+1,j+1})\, e_{\overline{\alpha}_j}. \end{aligned} \tag{7.1.20}$$

Therefore

$$\langle \overline{\alpha}_i, \overline{\alpha}_j \rangle = \frac{1}{2n}(2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1}), \quad \Big| \quad \text{simple root scalar products in } A_{n-1}. \tag{7.1.21}$$

The Cartan matrix is defined as [Eqs. (5.3.8) and (5.3.9)]

$$A_{ij} = \frac{2\langle \overline{\alpha}_i, \overline{\alpha}_j \rangle}{\langle \overline{\alpha}_j, \overline{\alpha}_j \rangle} = -p_{\overline{\alpha}_i;\overline{\alpha}_j} (i \neq j), \tag{7.1.22}$$

where $p_{\overline{\alpha}_i;\overline{\alpha}_j} \in \{0, 1, 2, 3\}$ determines the number of states above $\overline{\alpha}_i$ in the root string generated by $e_{\overline{\alpha}_j}$. For A_{n-1}, Eq. (7.1.21) implies that

$$A_{ij} = 2\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1} \quad \Rightarrow \quad \hat{A} \to \begin{bmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 2 \end{bmatrix}, \quad \text{Cartan matrix for } A_{n-1}. \tag{7.1.23}$$

The corresponding Dynkin diagram is shown in Fig. 7.1. Because all roots have the same length, $\mathrm{su}(n) = A_{n-1}$ is simply laced.

Figure 7.1: $\mathrm{su}(n) = A_{n-1}$.

7.1.1 Example: su(4) $= A_3$.

Using the root-building algorithm from Sec. 5.3.2, we construct the roots for A_3.

1. At $k = 1$, we have the simple roots

$$\underset{(\overline{\alpha}_1)}{\begin{bmatrix}2 & -1 & 0\end{bmatrix}} \quad \underset{(\overline{\alpha}_2)}{\begin{bmatrix}-1 & 2 & -1\end{bmatrix}} \quad \underset{(\overline{\alpha}_3)}{\begin{bmatrix}0 & -1 & 2\end{bmatrix}} \tag{7.1.24a}$$

2. At $k = 2$, the sum $\rho_1^{(2)} \equiv \overline{\alpha}_1 + \overline{\alpha}_2$ is a valid root since [Eq. (7.1.22)]

$$\begin{aligned} p_{\overline{\alpha}_1;\overline{\alpha}_2} &= 1, \\ p_{\overline{\alpha}_2;\overline{\alpha}_1} &= 1. \end{aligned} \tag{7.1.24b}$$

The sum $\rho_2^{(2)} \equiv \overline{\alpha}_2 + \overline{\alpha}_3$ is a valid root since

$$\begin{aligned} p_{\overline{\alpha}_2;\overline{\alpha}_3} &= 1, \\ p_{\overline{\alpha}_3;\overline{\alpha}_2} &= 1. \end{aligned} \tag{7.1.24c}$$

The Dynkin labels for these level 2 roots are

$$\underset{\rho_1^{(2)}}{\begin{bmatrix}1 & 1 & -1\end{bmatrix}} \quad \underset{\rho_2^{(2)}}{\begin{bmatrix}-1 & 1 & 1\end{bmatrix}} \tag{7.1.24d}$$

The non-zero depths are

$$m_{\rho_1^{(2)};\overline{\alpha}_1} = m_{\rho_1^{(2)};\overline{\alpha}_2} = 1, \quad m_{\rho_2^{(2)};\overline{\alpha}_2} = m_{\rho_1^{(2)};\overline{\alpha}_3} = 1. \tag{7.1.24e}$$

3. At $k = 3$, we compute

$$\begin{aligned} p_{\rho_1^{(2)};\overline{\alpha}_1} &= m_{\rho_1^{(2)};\overline{\alpha}_1} - 1 = 0, \\ p_{\rho_1^{(2)};\overline{\alpha}_2} &= m_{\rho_1^{(2)};\overline{\alpha}_2} - 1 = 0, \\ p_{\rho_1^{(2)};\overline{\alpha}_2} &= m_{\rho_1^{(2)};\overline{\alpha}_2} + 1 = 1, \\ p_{\rho_2^{(2)};\overline{\alpha}_1} &= m_{\rho_1^{(2)};\overline{\alpha}_1} + 1 = 1, \\ p_{\rho_2^{(2)};\overline{\alpha}_2} &= m_{\rho_1^{(2)};\overline{\alpha}_2} - 1 = 0, \\ p_{\rho_2^{(2)};\overline{\alpha}_2} &= m_{\rho_1^{(2)};\overline{\alpha}_2} - 1 = 0. \end{aligned} \tag{7.1.24f}$$

Thus $\rho^{(3)} \equiv \rho_1^{(2)} + \overline{\alpha}_3 = \rho_2^{(2)} + \overline{\alpha}_1 = \overline{\alpha}_1 + \overline{\alpha}_2 + \overline{\alpha}_3$ is the only $k = 3$ root,

$$\underset{\rho^{(3)}}{\begin{bmatrix}1 & 0 & 1\end{bmatrix}} \tag{7.1.24g}$$

The depths are

$$m_{\rho^{(3)};\overline{\alpha}_1} = 1, \qquad m_{\rho^{(3)};\overline{\alpha}_2} = 0, \qquad m_{\rho^{(3)};\overline{\alpha}_3} = 1. \tag{7.1.24h}$$

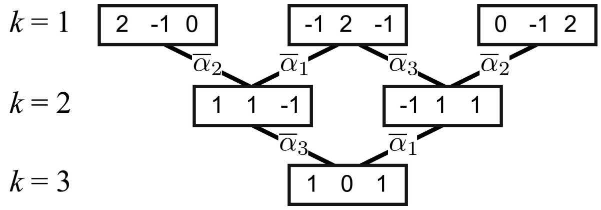


Figure 7.2: Root tree for su(4) $= A_3$. Roots are specified by their Dynkin labels. The level $k = 1$ simple roots appear at the top; linear combinations of these give the other positive roots. The labels on diagonal lines indicate the simple root that gives the level $(k+1)^{\text{th}}$ root from the k^{th} one.

4. At $k = 4$,

$$\begin{aligned} p_{\rho^{(3)};\overline{\alpha}_1} &= m_{\rho^{(3)};\overline{\alpha}_1} - 1 = 0, \\ p_{\rho^{(3)};\overline{\alpha}_2} &= m_{\rho^{(3)};\overline{\alpha}_2} - 0 = 0, p_{\rho^{(3)};\overline{\alpha}_3} = \quad m_{\rho^{(3)};\overline{\alpha}_3} - 1 = 0. \end{aligned} \tag{7.1.24i}$$

We conclude that there are no level $k = 4$ roots. $\rho^{(3)} = \overline{\alpha}_1 + \overline{\alpha}_2 + \overline{\alpha}_3 \equiv \theta$ is the highest root.

The results of the algorithm can be indicated with the branching diagram in Fig. 7.2. Since the roots are non-degenerate [Proposition (VI.), Sec. 5.2.2] and each positive root ρ has a corresponding negative root $-\rho$, we have a complete inventory for the set of roots Δ for A_3. There are $|\Delta| = 12$ roots, so the group dimension $d = 12 + 3 = 15$, as expected for su(4).

To get a geometrical picture of the algebra, we examine the Cartan matrix in Eq. (7.1.23). The angle between simple roots $\{\overline{\alpha}_1, \overline{\alpha}_2\}$ and between $\{\overline{\alpha}_2, \overline{\alpha}_3\}$ is 120°, while $\overline{\alpha}_1$ and $\overline{\alpha}_3$ are orthogonal. Combined with the fact that the algebra is simply laced and the inventory of roots (Fig. 7.2), one finds that the roots correspond to the twelve vertices of a certain quasiregular polyhedron (a cuboctahedron). See Fig. 7.3.

7.2 sp($2n$): C_n

Next we consider C_n, the Lie algebra associated to symplectic Sp($2n$) transformations. From module 2A, Eqs. (2A.3.1) and (2A.3.3), the defining conditions for a symplectic transformation $\hat{U}$ on a $2n$-component complex vector are

$$\hat{U}^\dagger \hat{U} = \hat{1}_{2n}, \tag{7.2.1a}$$

$$\hat{U}^\mathsf{T}\, \hat{\epsilon}\, \hat{U} = \hat{\epsilon}, \tag{7.2.1b}$$

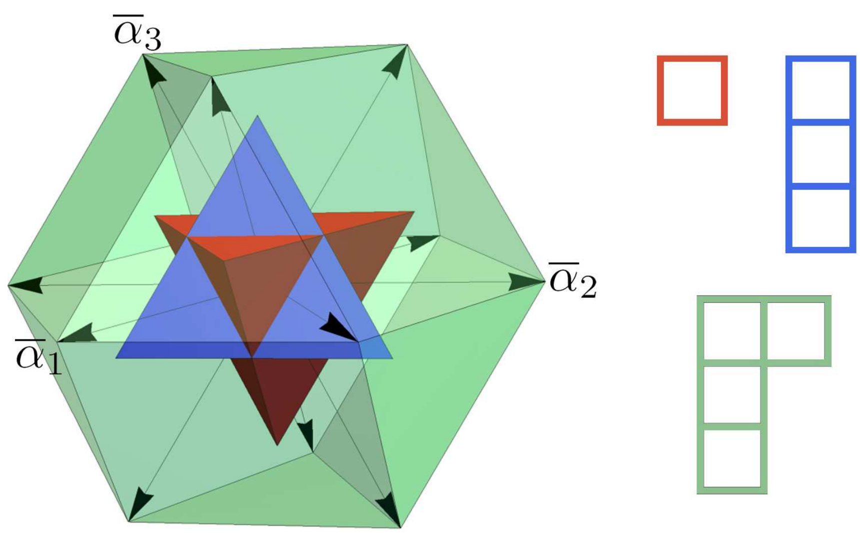


Figure 7.3: Root geometry for su(4) $= A_3$. The twelve roots correspond to the vertices of the green polyhedron (a cuboctahedron). The vertices of the embedded red (blue) tetrahedron correspond to the four weights of the defining "4" (conjugate "$\bar{4}$") representation, as demonstrated module 8. The box arrays on the right are the corresponding Young tableaux, also discussed in module 8.

where the $2n \times 2n$ matrix $\hat{\epsilon}$ is a block rank-2 Levi-Civita symbol,

$$\hat{\epsilon} \to \begin{bmatrix} 0 & \hat{1}_n \\ -\hat{1}_n & 0 \end{bmatrix}. \tag{7.2.2}$$

If we express $\hat{U}$ as the exponentiation of an (antihermitian) $2n \times 2n$ matrix $\hat{Y}$

$$\hat{U} = \exp(\hat{Y}),$$

then the symplectic condition translates to

$$\boxed{\hat{Y}^{\mathsf{T}} = \hat{\epsilon}\,\hat{Y}\,\hat{\epsilon}.} \tag{7.2.3}$$

As with $\hat{\epsilon}$, we decompose this into $n \times n$ blocks,

$$\hat{Y} = \begin{bmatrix} \hat{Y}_1 & \hat{Y}_2 \\ \hat{Y}_3 & \hat{Y}_4 \end{bmatrix}. \tag{7.2.4}$$

Eq. (7.2.3) then implies that

$$\begin{aligned} \hat{Y}_4 &= -\hat{Y}_1^{\mathsf{T}}, \\ \hat{Y}_2 &= \hat{Y}_2^{\mathsf{T}}, \\ \hat{Y}_3 &= \hat{Y}_3^{\mathsf{T}}. \end{aligned} \tag{7.2.5}$$

Let $1 \le j, k \le n$. We can define the following basis elements for $\hat{Y}$,

$$\hat{E}^1_{jk} \equiv \hat{E}_{jk} - \hat{E}_{k+n,j+n}, \tag{7.2.6a}$$
$$\hat{E}^2_{jk} \equiv \hat{E}_{j,k+n} + \hat{E}_{k,j+n} = \hat{E}^2_{(jk)}, \tag{7.2.6b}$$
$$\hat{E}^3_{jk} \equiv \hat{E}_{j+n,k} + \hat{E}_{k+n,j} = \hat{E}^3_{(jk)}. \tag{7.2.6c}$$

The parentheses here denote symmetrization [Eq. (1.4.2)]. The total number of such elements is

$$n^2 + n(n+1) = n(2n+1),$$

which is the correct group dimension [Eq. (2A.3.5)].

Explicitly for $n = 5$ [sp(10)], we have (e.g.)

$$\hat{E}^1_{34} \to \left[\begin{array}{ccccc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{7.2.7a}$$

$$\hat{E}^2_{34} \to \left[\begin{array}{ccccc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{7.2.7b}$$

$$\hat{E}^3_{25} \rightarrow \left[\begin{array}{ccccc|ccccc} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0 \end{array}\right]. \tag{7.2.7c}$$

Using Eq. (7.1.3), one can verify the following Lie brackets implied by the corresponding commutation relations:

$$[e^1_{jk}, e^1_{lp}] = \delta_{kl}\, e^1_{jp} - \delta_{jp}\, e^1_{lk}, \tag{7.2.8a}$$
$$[e^1_{jk}, e^2_{lp}] = \delta_{kl}\, e^2_{jp} + \delta_{pk}\, e^2_{jl}, \tag{7.2.8b}$$
$$[e^1_{jk}, e^3_{lp}] = -\,\delta_{jp}\, e^3_{kl} - \delta_{jl}\, e^3_{kp}, \tag{7.2.8c}$$
$$[e^2_{jk}, e^2_{lp}] = 0, \tag{7.2.8d}$$
$$[e^3_{jk}, e^3_{lp}] = 0, \tag{7.2.8e}$$
$$[e^2_{jk}, e^3_{lp}] = \delta_{kl}\, e^1_{jp} + \delta_{jp}\, e^1_{kl} + \delta_{pk}\, e^1_{jl} + \delta_{jl}\, e^1_{kp}. \tag{7.2.8f}$$

• **Exercise**: Verify all of the Lie brackets in Eq. (7.2.8).

A basis for the n-dimensional Cartan subalgebra H is given by the diagonal traceless elements $\{e^1_{jj}\}$, $1 \le j \le n$; a generic element $h \in H$ is

$$h = \sum_{j=1}^{n} \lambda_j\, e^1_{jj}. \tag{7.2.9}$$

Then

$$[h, e^1_{jk}] = (\lambda_j - \lambda_k)\, e^1_{jk}, \tag{7.2.10a}$$
$$[h, e^2_{jk}] = (\lambda_j + \lambda_k)\, e^2_{jk}, \tag{7.2.10b}$$
$$[h, e^3_{jk}] = -\,(\lambda_j + \lambda_k)\, e^3_{jk}, \tag{7.2.10c}$$

so that the collection of $\{e^{1,2,3}_{jk}\}$ excluding diagonal $\{e^1_{jj}\}$ is the set of root vectors.

The simple roots can be associated to the root vectors

(7.2.11)

root vector	root $\alpha(h)$
$e^1_{12} \equiv e_{\overline{\alpha}_1}$	$\overline{\alpha}_1(h) = \lambda_1 - \lambda_2$
$e^1_{23} \equiv e_{\overline{\alpha}_2}$	$\overline{\alpha}_2(h) = \lambda_2 - \lambda_3$
$\vdots$	$\vdots$
$e^1_{n-1,n} \equiv e_{\overline{\alpha}_{n-1}}$	$\overline{\alpha}_{n-1}(h) = \lambda_{n-1} - \lambda_n$
$e^2_{n,n} \equiv e_{\overline{\alpha}_n}$	$\overline{\alpha}_n(h) = 2\lambda_n.$

By definition, we take $\overline{\alpha}_1 > \overline{\alpha}_2 > \ldots > \overline{\alpha}_n$ (lexicographic ordering). Then positive roots are associated to root vectors $\{e^1_{jk}\}$ with $j < k$, and to $\{e^2_{jk}\}$. Negative root vectors are $\{e^1_{jk}\}$ $(j > k)$ and $\{e^3_{jk}\}$. The positive root associated to e^1_{jk} $(j < k)$ is expressed in terms of simple roots identically as in Eq. (7.1.11). For e^2_{ij}, the decomposition is (taking $j \ge i$ WLoG)

$$\lambda_i + \lambda_j = (\lambda_i - \lambda_j) + 2(\lambda_j - \lambda_n) + 2\lambda_n = \sum_{l=i}^{j-1} \overline{\alpha}_l + 2\sum_{l=j}^{n-1} \overline{\alpha}_l + \overline{\alpha}_n. \tag{7.2.12}$$

The commutation relations between the simple root vectors are given by

$$[e_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] = \delta_{j,i+1}\, e^1_{i,i+2} - \delta_{j,i-1}\, e^1_{i-1,i+1}, \qquad 1 \le i, j \le n-1, \tag{7.2.13a}$$
$$[e_{\overline{\alpha}_i}, e_{\overline{\alpha}_n}] = 2\delta_{i,n-1}\, e^2_{n-1,n}, \qquad 1 \le i \le n-1, \tag{7.2.13b}$$
$$[e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_j}] = \delta_{ij}\,\left(e^1_{i,i} - e^1_{i+1,i+1}\right), \qquad 1 \le i, j \le n-1, \tag{7.2.13c}$$
$$[e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_n}] = 0, \qquad 1 \le i \le n-1, \tag{7.2.13d}$$
$$[e_{\overline{\alpha}_n}, e_{-\overline{\alpha}_n}] = 4\, e^1_{nn}. \tag{7.2.13e}$$

The Killing form between Cartan subalgebra elements

$$h \equiv \sum_i \lambda_i e^1_{ii}, \quad h' \equiv \sum_j \lambda'_j e^1_{jj}, \tag{7.2.14}$$

is given by

$$\begin{aligned}
(h, h') &= \sum_{\substack{p,q=1\\p\neq q}}^n \Big([h,[h', e^1_{pq}]]\Big)_{\text{coeff. of } e^1_{pq}} + \sum_{p\ge q=1}^n \Big([h,[h', e^2_{pq}]]\Big)_{\text{coeff. of } e^2_{pq}} + \sum_{p\ge q=1}^n \Big([h,[h', e^3_{pq}]]\Big)_{\text{coeff. of } e^3_{pq}} \\
&= \sum_{p,q=1}^n (\lambda_p - \lambda_q)(\lambda'_p - \lambda'_q) + 2\sum_{p\ge q=1}^n (\lambda_p + \lambda_q)(\lambda'_p + \lambda'_q) \\
&= \sum_{p,q=1}^n (\lambda_p - \lambda_q)(\lambda'_p - \lambda'_q) + \sum_{p,q=1}^n (\lambda_p + \lambda_q)(\lambda'_p + \lambda'_q) + 4\sum_{p=1}^n \lambda_p \lambda'_p \\
&= 4(n+1)\sum_{p=1}^n \lambda_p \lambda'_p.
\end{aligned} \tag{7.2.15}$$

Now since

$$(h_{\overline{\alpha}_i}, h') = \overline{\alpha}_i(h') = \begin{cases} \lambda'_i - \lambda'_{i+1}, & 1 \le i \le n-1, \\ 2\lambda'_n, & i = n, \end{cases} \tag{7.2.16}$$

we conclude that

$$\begin{aligned}
h_{\overline{\alpha}_i} &= \frac{1}{4(n+1)}\left(e^1_{ii} - e^1_{i+1,i+1}\right) \equiv \sum_{j=1}^n \lambda_j^{(\overline{\alpha}_i)} e^1_{jj}, \qquad \lambda_j^{(\overline{\alpha}_i)} = \frac{1}{4(n+1)}\left(\delta_{i,j} - \delta_{i+1,j}\right), \qquad 1 \le i \le n-1, \\
h_{\overline{\alpha}_n} &= \frac{1}{2(n+1)} e^1_{nn} \equiv \sum_{j=1}^n \lambda_j^{(\overline{\alpha}_n)} e^1_{jj}, \qquad \lambda_j^{(\overline{\alpha}_n)} = \frac{1}{2(n+1)}\delta_{j,n}.
\end{aligned} \tag{7.2.17}$$

Eqs. (7.2.13c), (7.2.13e), (7.2.17) and (7.1.18) therefore give the Killing forms of the simple root vectors,

$$\begin{aligned}
(e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_i}) &= 4(n+1), \qquad 1 \le i \le n-1, \\
(e_{\overline{\alpha}_n}, e_{-\overline{\alpha}_n}) &= 8(n+1).
\end{aligned} \tag{7.2.18}$$

We can now compute the scalar products between simple roots, using

$$\begin{aligned}
[h_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] &= \langle \overline{\alpha}_i, \overline{\alpha}_j \rangle e_{\overline{\alpha}_j} \\
&= \left(\lambda_j^{(\overline{\alpha}_i)} - \lambda_{j+1}^{(\overline{\alpha}_i)}\right) e_{\overline{\alpha}_j} = \frac{1}{4(n+1)}\left(2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1}\right) e_{\overline{\alpha}_j}, \qquad 1 \le i, j \le n-1, \\
[h_{\overline{\alpha}_n}, e_{\overline{\alpha}_j}] &= \langle \overline{\alpha}_n, \overline{\alpha}_j \rangle e_{\overline{\alpha}_j} \\
&= \left(\lambda_j^{(\overline{\alpha}_n)} - \lambda_{j+1}^{(\overline{\alpha}_n)}\right) e_{\overline{\alpha}_j} = -\frac{1}{2(n+1)}\,\delta_{j+1,n}\, e_{\overline{\alpha}_j}, \qquad 1 \le j \le n-1, \\
[h_{\overline{\alpha}_n}, e_{\overline{\alpha}_n}] &= \langle \overline{\alpha}_n, \overline{\alpha}_n \rangle e_{\overline{\alpha}_n} \\
&= \frac{1}{(n+1)} e_{\overline{\alpha}_n}.
\end{aligned} \tag{7.2.19}$$

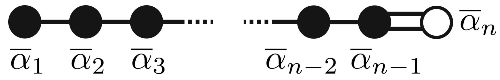


Figure 7.4: $\mathrm{sp}(2n) = C_n$.

Therefore

$$\begin{aligned}
\langle \overline{\alpha}_i, \overline{\alpha}_j \rangle &= \frac{1}{4(n+1)} \left(2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1} \right), \quad 1 \le i, j \le n-1, \\
\langle \overline{\alpha}_i, \overline{\alpha}_n \rangle &= -\frac{1}{2(n+1)} \delta_{i,n-1}, \quad 1 \le i \le n-1, \\
\langle \overline{\alpha}_n, \overline{\alpha}_n \rangle &= \frac{1}{(n+1)},
\end{aligned} \qquad \text{simple root scalar products in } C_n. \tag{7.2.20}$$

The last equation implies that $\overline{\alpha}_n$ is a long root, with $|\overline{\alpha}_n|^2 = 2|\overline{\alpha}_j|^2$, $1 \le j \le n-1$. C_n is not simply laced.

The corresponding Cartan matrix is given by [via Eq. (7.1.22)]

$$\hat{A} \rightarrow \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -2 & 2 \end{bmatrix}, \qquad \text{Cartan matrix for } C_n. \tag{7.2.21}$$

The Dynkin diagram is shown in Fig. 7.4.

7.3 so($2n$): D_n

Next we consider D_n, the Lie algebra associated to special orthogonal SO($2n$) transformations on even-dimensional vectors. From Eq. (2A.2.2) in module 2A, the defining condition for an orthogonal transformation $\hat{U}$ on a $2n$-component vector is

$$\hat{U}^{\mathsf{T}} \hat{U} = \hat{1}_{2n}. \tag{7.3.1}$$

If we express $\hat{U}$ as the exponentiation of an (antihermitian) $2n \times 2n$ matrix $\hat{Y}$

$$\hat{U} = \exp(\hat{Y}),$$

then we have the antisymmetry condition

$$-\hat{Y}^{\mathsf{T}} = \hat{Y}. \tag{7.3.2}$$

Since we would like to identify Cartan subalgebra elements with diagonal matrices, Eq. (7.3.2) is inconvenient. Instead, we make a unitary (but *not* orthogonal) basis transformation,

$$\hat{U}' \equiv \hat{V}^\dagger \, \hat{U} \, \hat{V}, \qquad \hat{V}^\dagger \hat{V} = \hat{1}_{2n}. \tag{7.3.3}$$

Then

$$\begin{aligned}
&\hat{U}'^{\mathsf{T}} \hat{M} \hat{U}' = \hat{M}, \\
&\hat{M} \equiv \hat{V}^{\mathsf{T}} \hat{V} = \hat{M}^{\mathsf{T}}, \quad \hat{M}^\dagger \hat{M} = \hat{1}_{2n}.
\end{aligned} \tag{7.3.4}$$

For $\hat{U}' = \exp(\hat{Y}')$,

$$-\hat{M}^\dagger \, \hat{Y}'^{\mathsf{T}} \, \hat{M} = \hat{Y}'. \tag{7.3.5}$$

We choose $\hat{V}$ to have the block form

$$\hat{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} i\hat{1}_n & -i\hat{1}_n \\ -\hat{1}_n & -\hat{1}_n \end{bmatrix}, \quad \Rightarrow \quad \hat{M} = \hat{M}^\dagger = \begin{bmatrix} 0 & \hat{1}_n \\ \hat{1}_n & 0 \end{bmatrix}. \tag{7.3.6}$$

As with $\hat{M}$, we decompose $\hat{Y'}$ into $n \times n$ blocks,

$$\hat{Y}' = \begin{bmatrix} \hat{Y}_1 & \hat{Y}_2 \\ \hat{Y}_3 & \hat{Y}_4 \end{bmatrix}. \tag{7.3.7}$$

Eq. (7.3.5) then implies that

$$\begin{aligned} \hat{Y}_4 &= -\hat{Y}_1^\mathsf{T}, \\ \hat{Y}_2 &= -\hat{Y}_2^\mathsf{T}, \\ \hat{Y}_3 &= -\hat{Y}_3^\mathsf{T}. \end{aligned} \tag{7.3.8}$$

This is almost the same as the symplectic case [Eq. (7.2.5)], except the off-diagonal blocks are now antisymmetric.

Let $1 \le j, k \le n$. We can define the following basis elements for $\hat{Y}$,

$$\hat{E}^1_{jk} \equiv \hat{E}_{jk} - \hat{E}_{k+n,j+n}, \tag{7.3.9a}$$
$$\hat{E}^2_{jk} \equiv \hat{E}_{j,k+n} - \hat{E}_{k,j+n} = \hat{E}^2_{[jk]}, \tag{7.3.9b}$$
$$\hat{E}^3_{jk} \equiv \hat{E}_{j+n,k} - \hat{E}_{k+n,j} = \hat{E}^3_{[jk]}. \tag{7.3.9c}$$

The parentheses here denote antisymmetrization [Eq. (1.4.2)]. The total number of such elements is

$$n^2 + n(n-1) = n(2n-1),$$

which is the correct group dimension [Eq. (7.3.2)].

Relative to the $n = 5$ symplectic example in Eq. (7.2.7),

$$\hat{E}^2_{34} \to \left[\begin{array}{ccccc|ccccc} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&-1&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{array}\right], \tag{7.3.10a}$$

$$\hat{E}^3_{25} \to \left[\begin{array}{ccccc|ccccc} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&-1&0&0&0&0&0&0&0&0 \end{array}\right]. \tag{7.3.10b}$$

Using Eq. (7.1.3), the Lie brackets are given by [c.f. Eq. (7.2.8)]

$$[e^1_{jk}, e^1_{lp}] = \delta_{kl}\, e^1_{jp} - \delta_{jp}\, e^1_{lk}, \quad (7.3.11a)$$
$$[e^1_{jk}, e^2_{lp}] = \delta_{kl}\, e^2_{jp} - \delta_{pk}\, e^2_{jl}, \quad (7.3.11b)$$
$$[e^1_{jk}, e^3_{lp}] = \delta_{jp}\, e^3_{kl} - \delta_{jl}\, e^3_{kp}, \quad (7.3.11c)$$
$$[e^2_{jk}, e^2_{lp}] = 0, \quad (7.3.11d)$$
$$[e^3_{jk}, e^3_{lp}] = 0, \quad (7.3.11e)$$
$$[e^2_{jk}, e^3_{lp}] = \delta_{kl}\, e^1_{jp} + \delta_{jp}\, e^1_{kl} - \delta_{pk}\, e^1_{jl} - \delta_{jl}\, e^1_{kp}. \quad (7.3.11f)$$

• **Exercise**: Verify the Lie brackets in Eq. (7.3.11).

A generic Cartan subalgebra H element is given by

$$h = \sum_{j=1}^{n} \lambda_j\, e^1_{jj}. \quad (7.3.12)$$

Then

$$[h, e^1_{jk}] = (\lambda_j - \lambda_k)\, e^1_{jk}, \quad (7.3.13a)$$
$$[h, e^2_{jk}] = (\lambda_j + \lambda_k)\, e^2_{jk}, \quad (7.3.13b)$$
$$[h, e^3_{jk}] = -(\lambda_j + \lambda_k)\, e^3_{jk}. \quad (7.3.13c)$$

The simple roots can be associated to the root vectors

root vector	root $\alpha(h)$
$e^1_{12} \equiv e_{\overline{\alpha}_1}$	$\overline{\alpha}_1(h) = \lambda_1 - \lambda_2$
$e^1_{23} \equiv e_{\overline{\alpha}_2}$	$\overline{\alpha}_2(h) = \lambda_2 - \lambda_3$
$\vdots$	$\vdots$
$e^1_{n-1,n} \equiv e_{\overline{\alpha}_{n-1}}$	$\overline{\alpha}_{n-1}(h) = \lambda_{n-1} - \lambda_n$
$e^2_{n-1,n} \equiv e_{\overline{\alpha}_n}$	$\overline{\alpha}_n(h) = \lambda_{n-1} + \lambda_n.$

(7.3.14)

By definition, we take $\overline{\alpha}_1 > \overline{\alpha}_2 > \ldots > \overline{\alpha}_n$ (lexicographic ordering). Then positive roots are associated to root vectors $\{e^1_{jk}\}$ with $j < k$, and to $\{e^2_{jk}\}$. Negative root vectors are $\{e^1_{jk}\}$ $(j > k)$ and $\{e^3_{jk}\}$. The positive root associated to e^1_{jk} $(j < k)$ is expressed in terms of simple roots identically as in Eq. (7.1.11). For e^2_{ij}, the decomposition is (taking $j > i$ WLoG)

$$\lambda_i + \lambda_j = (\lambda_i - \lambda_j) + 2(\lambda_j - \lambda_{n-1}) + (\lambda_{n-1} - \lambda_n) + (\lambda_{n-1} + \lambda_n) = \sum_{l=i}^{j-1} \overline{\alpha}_l + 2\sum_{l=j}^{n-2} \overline{\alpha}_l + \overline{\alpha}_{n-1} + \overline{\alpha}_n. \quad (7.3.15)$$

The commutation relations between the simple root vectors are given by

$$[e_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] = \delta_{j,i+1}\, e^1_{i,i+2} - \delta_{j,i-1}\, e^1_{i-1,i+1}, \qquad 1 \le i, j \le n-1, \quad (7.3.16a)$$
$$[e_{\overline{\alpha}_i}, e_{\overline{\alpha}_n}] = \delta_{i,n-2}\, e^2_{n-2,n}, \qquad 1 \le i \le n-1, \quad (7.3.16b)$$
$$[e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_j}] = \delta_{ij}\, \left(e^1_{i,i} - e^1_{i+1,i+1}\right), \qquad 1 \le i, j \le n-1, \quad (7.3.16c)$$
$$[e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_n}] = 0, \qquad 1 \le i \le n-1, \quad (7.3.16d)$$
$$[e_{\overline{\alpha}_n}, e_{-\overline{\alpha}_n}] = e^1_{n-1,n-1} + e^1_{n,n}. \quad (7.3.16e)$$

The Killing form between Cartan subalgebra elements

$$h \equiv \sum_i \lambda_i e^1_{ii}, \quad h' \equiv \sum_j \lambda'_j e^1_{jj}, \quad (7.3.17)$$

is given by

$$(h, h') = 4(n-1)\sum_{p=1}^{n} \lambda_p \lambda'_p. \quad (7.3.18)$$

• **Exercise**: Verify Eq. (7.3.18).

Now since

$$(h_{\overline{\alpha}_i}, h') = \overline{\alpha}_i(h') = \begin{cases} \lambda'_i - \lambda'_{i+1}, & 1 \le i \le n-1, \\ \lambda'_{n-1} + \lambda'_n, & i = n, \end{cases} \tag{7.3.19}$$

we conclude that

$$\begin{aligned} h_{\overline{\alpha}_i} &= \frac{1}{4(n-1)} \left(e^1_{ii} - e^1_{i+1,i+1} \right) \equiv \sum_{j=1}^n \lambda_j^{(\overline{\alpha}_i)} e^1_{jj}, \qquad \lambda_j^{(\overline{\alpha}_i)} = \frac{1}{4(n-1)} \left(\delta_{i,j} - \delta_{i+1,j} \right), \qquad 1 \le i \le n-1, \\ h_{\overline{\alpha}_n} &= \frac{1}{4(n-1)} \left(e^1_{n-1,n-1} + e^1_{n,n} \right) \equiv \sum_{j=1}^n \lambda_j^{(\overline{\alpha}_n)} e^1_{jj}, \qquad \lambda_j^{(\overline{\alpha}_n)} = \frac{1}{4(n-1)} \left(\delta_{n-1,j} + \delta_{n,j} \right). \end{aligned} \tag{7.3.20}$$

Eqs. (7.3.16c), (7.3.16e), (7.3.20) and (7.1.18) therefore give the Killing forms of the simple root vectors,

$$(e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_i}) = 4(n-1), \qquad 1 \le i \le n. \tag{7.3.21}$$

We can now compute the scalar products between simple roots, using

$$\begin{aligned} [h_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] &= \frac{1}{4(n-1)} \left(2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1} \right) e_{\overline{\alpha}_j}, \qquad 1 \le i,j \le n-1, \\ [h_{\overline{\alpha}_n}, e_{\overline{\alpha}_j}] &= -\frac{1}{4(n-1)} \delta_{j,n-2}\, e_{\overline{\alpha}_j}, \qquad 1 \le j \le n-1, \\ [h_{\overline{\alpha}_n}, e_{\overline{\alpha}_n}] &= \frac{2}{4(n-1)} e_{\overline{\alpha}_n}. \end{aligned} \tag{7.3.22}$$

Therefore

$$\left. \begin{aligned} \langle \overline{\alpha}_i, \overline{\alpha}_j \rangle &= \frac{1}{4(n-1)} \left(2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1} \right), \quad 1 \le i,j \le n-1, \\ \langle \overline{\alpha}_i, \overline{\alpha}_n \rangle &= -\frac{1}{4(n-1)} \delta_{i,n-2}, \qquad 1 \le i \le n-1, \\ \langle \overline{\alpha}_n, \overline{\alpha}_n \rangle &= \frac{2}{4(n-1)}, \end{aligned} \right| \quad \text{simple root scalar products in } D_n. \tag{7.3.23}$$

Since all roots have the same length, D_n is simply laced.

The corresponding Cartan matrix is given by [via Eq. (7.1.22)]

$$\hat{A} \to \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 & 2 \end{bmatrix}, \quad \text{Cartan matrix for } D_n. \tag{7.3.24}$$

The Dynkin diagram is shown in Fig. 7.5.

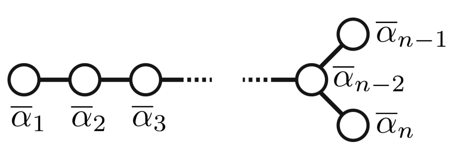


Figure 7.5: so$(2n) = D_n$.

7.4 so$(2n+1)$: B_n

Finally we consider $B_n \Leftrightarrow$ so$(2n+1)$. We can view the B_n generators in the defining representation as equivalent to those for D_n, but with an added "zeroth" row and "zeroth" column preceding the $2n \times 2n$ generators of so$(2n)$; i.e. a vector index i runs from $0 \leq i \leq 2n$.

As in Sec. 7.3, we begin by rotating the orthogonal condition $\hat{U}^{\mathsf{T}}\hat{U} = \hat{1}_{2n+1}$ to a different basis. Applying the transformation as in Eq. (7.3.3), we choose

$$\hat{V} = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & \hat{0}_{1,n} & \hat{0}_{1,n} \\ \hat{0}_{n,1} & i\hat{1}_n & -i\hat{1}_n \\ \hat{0}_{n,1} & -\hat{1}_n & -\hat{1}_n \end{bmatrix}, \quad \Rightarrow \quad \hat{M} = \hat{V}^{\mathsf{T}}\hat{V} = \begin{bmatrix} 1 & \hat{0}_{1,n} & \hat{0}_{1,n} \\ \hat{0}_{n,1} & 0 & \hat{1}_n \\ \hat{0}_{n,1} & \hat{1}_n & 0 \end{bmatrix}. \tag{7.4.1}$$

Here $\hat{0}_{1,n}$ ($\hat{0}_{n,1}$) denotes an n-fold row (column) of zeroes. The $(2n+1)\times(2n+1)$ generator $\hat{Y}'$ satisfies the condition as in Eq. (7.3.5), which allows the block decomposition [c.f. Eqs. (7.3.7) and (7.3.8)]

$$\hat{Y}' = \begin{bmatrix} b^1 & \hat{c}^1_{1,n} & \hat{c}^2_{1,n} \\ \hat{d}^1_{n,1} & \hat{Y}_1 & \hat{Y}_2 \\ \hat{d}^2_{n,1} & \hat{Y}_3 & \hat{Y}_4 \end{bmatrix}. \tag{7.4.2}$$

Eq. (7.3.5) then implies that

$$\begin{aligned} \hat{Y}_4 =& -\hat{Y}_1^{\mathsf{T}}, \\ \hat{Y}_2 =& -\hat{Y}_2^{\mathsf{T}}, \\ \hat{Y}_3 =& -\hat{Y}_3^{\mathsf{T}}, \\ b^1 =& 0, \\ \hat{c}^1_{1,n} =& -\left[\hat{d}^2_{n,1}\right]^{\mathsf{T}}, \\ \hat{c}^2_{1,n} =& -\left[\hat{d}^1_{n,1}\right]^{\mathsf{T}}. \end{aligned} \tag{7.4.3}$$

The basis elements for $\hat{Y}$ include $\{\hat{E}^1_{jk}, \hat{E}^2_{jk}, \hat{E}^3_{jk}\}$ ($1 \leq j,k \leq n$) from Eq. (7.3.9), and

$$\hat{E}^4_j \equiv \hat{E}_{0,j} - \hat{E}_{j+n,0}, \tag{7.4.4a}$$
$$\hat{E}^5_j \equiv \hat{E}_{j,0} - \hat{E}_{0,j+n}, \tag{7.4.4b}$$

both with $1 \leq j \leq n$. The total number of such elements is

$$n(2n-1) + 2n = n(2n+1).$$

Relative to the example in Eq. (7.3.10), $\{\hat{E}^1_{jk}, \hat{E}^2_{jk}, \hat{E}^3_{jk}\}$ are now block embedded in 11×11 matrices (after one initial row and one initial column of zeroes), while (e.g.)

$$\hat{E}^4_4 \rightarrow \left[\begin{array}{c|ccccc|ccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{7.4.5a}$$

$$\hat{E}_3^5 \rightarrow \left[\begin{array}{c|ccccc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{7.4.5b}$$

In addition to the Lie brackets in Eq. (7.3.11), we have

$$[e_{jk}^1, e_l^4] = -\,\delta_{jl}\, e_k^4, \tag{7.4.6a}$$
$$[e_{jk}^1, e_l^5] = \delta_{kl}\, e_j^5, \tag{7.4.6b}$$
$$[e_{jk}^2, e_l^4] = \delta_{jl}\, e_k^5 - \delta_{kl}\, e_j^5, \tag{7.4.6c}$$
$$[e_{jk}^2, e_l^5] = 0, \tag{7.4.6d}$$
$$[e_{jk}^3, e_l^4] = 0, \tag{7.4.6e}$$
$$[e_{jk}^3, e_l^5] = \delta_{jl}\, e_k^4 - \delta_{kl}\, e_j^4, \tag{7.4.6f}$$
$$[e_j^4, e_k^4] = -\,e_{jk}^3, \tag{7.4.6g}$$
$$[e_j^5, e_k^5] = -\,e_{jk}^2, \tag{7.4.6h}$$
$$[e_j^4, e_k^5] = -\,e_{kj}^1. \tag{7.4.6i}$$

• **Exercise**: Verify the Lie brackets in Eq. (7.4.6).

A generic Cartan subalgebra H element is given by

$$h = \sum_{j=1}^{n} \lambda_j\, e_{jj}^1. \tag{7.4.7}$$

Then

$$[h, e_{jk}^1] = (\lambda_j - \lambda_k)\, e_{jk}^1, \tag{7.4.8a}$$
$$[h, e_{jk}^2] = (\lambda_j + \lambda_k)\, e_{jk}^2, \tag{7.4.8b}$$
$$[h, e_{jk}^3] = -\,(\lambda_j + \lambda_k)\, e_{jk}^3, \tag{7.4.8c}$$
$$[h, e_j^4] = -\,\lambda_j\, e_j^4, \tag{7.4.8d}$$
$$[h, e_j^5] = \lambda_j e_j^5. \tag{7.4.8e}$$

The simple roots can be associated to the root vectors

root vector	root $\alpha(h)$
$e_{12}^1 \equiv e_{\overline{\alpha}_1}$	$\overline{\alpha}_1(h) = \lambda_1 - \lambda_2$
$e_{23}^1 \equiv e_{\overline{\alpha}_2}$	$\overline{\alpha}_2(h) = \lambda_2 - \lambda_3$
$\vdots$	$\vdots$
$e_{n-1,n}^1 \equiv e_{\overline{\alpha}_{n-1}}$	$\overline{\alpha}_{n-1}(h) = \lambda_{n-1} - \lambda_n$
$e_n^5 \equiv e_{\overline{\alpha}_n}$	$\overline{\alpha}_n(h) = \lambda_n.$

(7.4.9)

By definition, we take $\overline{\alpha}_1 > \overline{\alpha}_2 > \ldots > \overline{\alpha}_n$ (lexicographic ordering). The positive roots are associated to root vectors $\{e_{jk}^1\}$ with $j < k$, $\{e_{jk}^2\}$, and $\{e_j^5\}$. Negative root vectors are $\{e_{jk}^1\}$ $(j > k)$, $\{e_{jk}^3\}$, and $\{e_j^4\}$. The positive root associated to e_{jk}^1 $(j < k)$ is expressed in terms of simple roots identically as in Eq. (7.1.11). For e_{ij}^2, the decomposition is (taking $j > i$ WLoG)

$$\lambda_i + \lambda_j = (\lambda_i - \lambda_j) + 2(\lambda_j - \lambda_n) + 2\lambda_n = \sum_{l=i}^{j-1} \overline{\alpha}_l + 2\sum_{l=j}^{n-1} \overline{\alpha}_l + 2\overline{\alpha}_n. \tag{7.4.10}$$

Eq. (7.4.10) has *almost* the same form as the symplectic C_n case, Eq. (7.2.12); the difference is the prefactor of two for $\overline{\alpha}_n$ in Eq. (7.4.10). In fact we will see that B_n and C_n are closely related (although not identical, except for $B_2 = C_2$). For e_i^5, the decomposition is

$$\lambda_i = (\lambda_i - \lambda_n) + \lambda_n = \sum_{l=i}^{n-1} \overline{\alpha}_l + \overline{\alpha}_n. \tag{7.4.11}$$

The commutation relations between the simple root vectors are given by

$$\begin{aligned}
[e_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] &= \delta_{j,i+1}\, e^1_{i,i+2} - \delta_{j,i-1}\, e^1_{i-1,i+1}, && 1 \le i,j \le n-1, && (7.4.12a)\\
[e_{\overline{\alpha}_i}, e_{\overline{\alpha}_n}] &= \delta_{i,n-1}\, e^5_{n-1}, && 1 \le i \le n-1, && (7.4.12b)\\
[e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_j}] &= \delta_{ij}\, \left(e^1_{i,i} - e^1_{i+1,i+1}\right), && 1 \le i,j \le n-1, && (7.4.12c)\\
[e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_n}] &= 0, && 1 \le i \le n-1, && (7.4.12d)\\
[e_{\overline{\alpha}_n}, e_{-\overline{\alpha}_n}] &= e^1_{nn}. && && (7.4.12e)
\end{aligned}$$

The Killing form between Cartan subalgebra elements

$$h \equiv \sum_i \lambda_i e^1_{ii}, \quad h' \equiv \sum_j \lambda'_j e^1_{jj}, \tag{7.4.13}$$

is given by

$$(h, h') = 4\left(n - \frac{1}{2}\right) \sum_{p=1}^{n} \lambda_p \lambda'_p. \tag{7.4.14}$$

• **Exercise**: Verify Eq. (7.4.14).

Now since

$$(h_{\overline{\alpha}_i}, h') = \overline{\alpha}_i(h') = \begin{cases} \lambda'_i - \lambda'_{i+1}, & 1 \le i \le n-1, \\ \lambda'_n, & i = n, \end{cases} \tag{7.4.15}$$

we conclude that

$$\begin{aligned}
h_{\overline{\alpha}_i} &= \frac{1}{4\left(n-\frac{1}{2}\right)} \left(e^1_{ii} - e^1_{i+1,i+1}\right) \equiv \sum_{j=1}^{n} \lambda_j^{(\overline{\alpha}_i)} e^1_{jj}, \qquad \lambda_j^{(\overline{\alpha}_i)} = \frac{1}{4\left(n-\frac{1}{2}\right)} \left(\delta_{i,j} - \delta_{i+1,j}\right), \qquad 1 \le i \le n-1,\\
h_{\overline{\alpha}_n} &= \frac{1}{4\left(n-\frac{1}{2}\right)} e^1_{n,n} \equiv \sum_{j=1}^{n} \lambda_j^{(\overline{\alpha}_n)} e^1_{jj}, \qquad \lambda_j^{(\overline{\alpha}_n)} = \frac{1}{4\left(n-\frac{1}{2}\right)} \delta_{n,j}.
\end{aligned} \tag{7.4.16}$$

Eqs. (7.4.12c), (7.4.12e), (7.4.16) and (7.1.18) therefore give the Killing forms of the simple root vectors,

$$(e_{\overline{\alpha}_i}, e_{-\overline{\alpha}_i}) = 4\left(n - \frac{1}{2}\right), \qquad 1 \le i \le n. \tag{7.4.17}$$

We can now compute the scalar products between simple roots, using

$$\begin{aligned}
[h_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] &= \frac{1}{4\left(n-\frac{1}{2}\right)} \left(2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1}\right) e_{\overline{\alpha}_j}, \qquad 1 \le i,j \le n-1,\\
[h_{\overline{\alpha}_n}, e_{\overline{\alpha}_j}] &= -\frac{1}{4\left(n-\frac{1}{2}\right)} \delta_{j,n-1}\, e_{\overline{\alpha}_j}, \qquad 1 \le j \le n-1,\\
[h_{\overline{\alpha}_n}, e_{\overline{\alpha}_n}] &= \frac{1}{4\left(n-\frac{1}{2}\right)} e_{\overline{\alpha}_n}.
\end{aligned} \tag{7.4.18}$$

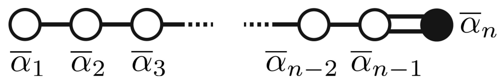


Figure 7.6: so$(2n+1) = B_n$.

Therefore

$$
\begin{aligned}
\langle \overline{\alpha}_i, \overline{\alpha}_j \rangle &= \frac{1}{4\left(n-\frac{1}{2}\right)} \left(2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1}\right), \quad 1 \le i, j \le n-1, \\
\langle \overline{\alpha}_i, \overline{\alpha}_n \rangle &= -\frac{1}{4\left(n-\frac{1}{2}\right)} \delta_{i,n-1}, \qquad 1 \le i \le n-1, \\
\langle \overline{\alpha}_n, \overline{\alpha}_n \rangle &= \frac{1}{4\left(n-\frac{1}{2}\right)}.
\end{aligned}
\qquad \text{simple root scalar products in } B_n. \tag{7.4.19}
$$

The last equation implies that $\overline{\alpha}_n$ is a short root, with $|\overline{\alpha}_n|^2 = |\overline{\alpha}_j|^2/2$, $1 \le j \le n-1$. Like C_n, B_n is not simply laced.

The corresponding Cartan matrix is given by [via Eq. (7.1.22)]

$$
\hat{A} \rightarrow \begin{bmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -2 \\ 0 & 0 & 0 & \ldots & -1 & 2 \end{bmatrix}, \qquad \text{Cartan matrix for } B_n. \tag{7.4.20}
$$

The Dynkin diagram is shown in Fig. 7.6.

The Cartan matrices/Dynkin diagrams for C_n [Eq. (7.2.21), Fig. 7.4] and B_n [Eq. (7.4.20), Fig. 7.6] are almost identical. In both cases, the only non-vanishing simple root inner products involve "nearest-neighbors" $\overline{\alpha}_i$ and $\overline{\alpha}_{i+1}$, $1 \le i \le n-1$. The first $n-1$ simple roots share the same length, while $\overline{\alpha}_n$ is a long (short) root for C_n (B_n). In fact, the root systems for B_n and C_n have the same geometry, except that long and short roots are interchanged.

On the other hand, we will see that D_n and B_n have more in common in terms of the n simplest (so called fundamental[1]) representations, described in module 8. In particular, both D_n and B_n possess fundamental spinor representations, which are qualitatively distinct and cannot be built by tensoring together defining vector indices. By contrast, every representation of C_n or A_n can be built by tensoring together indices transforming in the the defining $2n$-dimensional [$(n+1)$-dimensional] representation.

• **Exercise**: Find the inventory of roots and plot the 3D root geometry for B_3 and C_3, following the same approach used for A_3 in Sec. 7.1.1.

7.5 Classical Lie algebras to rank 4; equivalent algebras

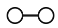
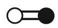
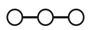
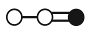
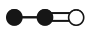
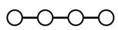
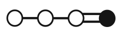
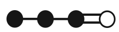
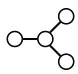
In Table 1, we show the classical Lie algebras through rank four. Various "accidental" equivalences between different algebras of the same rank are indicated, and easily recognized by the identical Dynkin diagrams. All four families are distinct for each rank above three. Of course, equivalence of the Lie algebras does not imply equivalence of the corresponding symmetry groups.

7.6 The exceptional algebras

The four classical families studied above do not exhaust the possibilities for (semi-)simple Lie algebras. It is perhaps surprising that there are only five additional possibilities corresponding to particular algebras of fixed rank. This can be shown by a careful determination of **allowable** Dynkin diagrams. The argument is explicated in chapter IX of Cahn [1] and chapter 20 of Georgi [2];

[1] So far we have used the terms "defining" and "fundamental" representations interchangably, as is common in physics. However, for a given Lie algebra L, there exists of *set* of "simplest" representations (which always includes the defining one), and mathematicians refer to each member of this set as a fundamental representation. We will define fundamental represenations precisely in module 8, using the **fundamental weights**.

Table 1: Classical Lie algebras to rank 4.

Rank	Dynkin	Cartan	Compact	# of positive roots
1		A_1	su(2) = sp(2) = so(3)	1
2		A_2	su(3)	3
		$B_2 = C_2$	so(5) = sp(4)	4
		$D_2 = A_1 \times A_1$	so(4) = su(2) × su(2)	2
3		$A_3 = D_3$	su(4) = so(6)	6
		B_3	so(7)	9
		C_3	sp(6)	9
4		A_4	su(5)	10
		B_4	so(9)	16
		C_4	sp(8)	16
		D_4	so(8)	12

we won't belabor it here. Instead, we simply state the Cartan matrices and Dynkin diagrams for the five exceptional algebras G_2, F_4, and $E_{6,7,8}$. Our labeling conventions are the same as chapter 13 of [3].

(7.6.1)

Cartan matrices for the exceptional Lie algebras G_2, F_4, E_6, E_7, and E_8

$$\hat{A}_{G_2} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}, \qquad \hat{A}_{E_6} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix},$$

$$\hat{A}_{F_4} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \qquad \hat{A}_{E_7} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{bmatrix}, \tag{7.6.2}$$

$$\hat{A}_{E_8} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}.$$

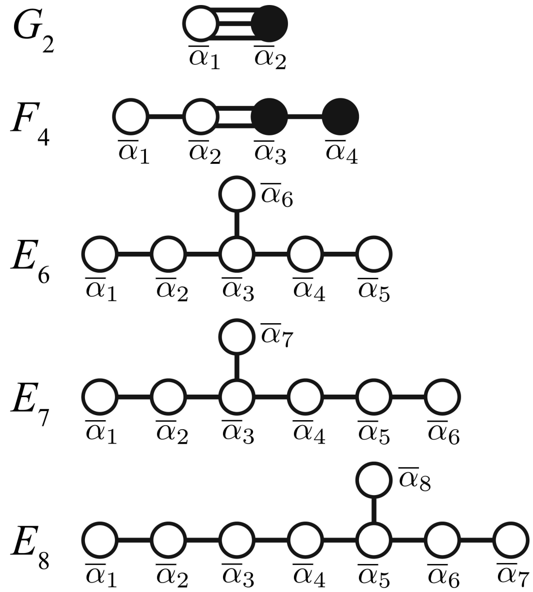


Figure 7.7: The five exceptional algebras.

References

[1] Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Benjamin/Cummings, Menlo Park, California, 1984).

[2] Howard Georgi, *Lie Algebras in Particle Physics*, 2nd ed. (Westview Press, Boulder, Colorado, 1999).

[3] Phillipe Di Francesco, Pierre Mathieu, David Sénéchal, *Conformal Field Theory* (Springer-Verlag, New York, 1996).