8B. Informal lecture: Some physics applications of highest weight representations
*** version 1.0 ***

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October 27, 2016

8B.1 Non-abelian gauge theory

- Gauge theory: couple a matter field (usually fermions) to a (4-vector) potential $A_\mu = \{A_0, \vec{A}\}$. In the Standard model, gauge bosons mediate the strong, weak, and electromagnetic forces.
- What is a gauge theory mathematically? Make a global symmetry local.
- Consider a theory with a global, continuous symmetry. Can be abelian [U(1) or O(2)] or non-abelian Lie group G.
- Under symmetry transformation, group acts on some "indices" carried by the matter field. E.g., the phase a complex fermion field, or the color of a quark field.
- Given the space in which the group acts, why should we need to make a global choice of coordinates? I.e., if I am measuring quark colors on earth and you measure them on the moon, why should we have to "synchronize" our coordinate frames. My green could be your blue.
- Formally, one can "gauge" a symmetry, introducing a gauge field $A_\mu(t, \mathbf{r})$ such that one can choose bases differently at all points in spacetime $(t, \mathbf{r})$. By itself, this doesn't introduce any physics, only a tremendous amount of mathematical redundancy.
- If all we are doing is choosing different bases, then all configurations of A_μ are "gauge equivalent" to zero. But there is another possibility. There can be gauge field configurations with some kind of gauge-invariant "curvature" present (e.g., magnetic or electric flux). In a quantum gauge theory, the physical gauge flux is carried by transverse-polarized gauge quanta (photons in QED, gluons in QCD, etc).

Simplest non-abelian example: Quantum chromodynamics (QCD), theory of quarks interacting via the strong force, carried by the gluons.

- Lagrangian density (fermion sector)

$$L = \bar{\psi}\left[\hat{\gamma}^\mu\left(i\frac{\partial}{\partial x^\mu} + g\hat{A}_\mu\right) - m\right]\psi = \bar{\psi}^{\sigma,i}\left\{(\hat{\gamma}^\mu)_{\sigma,\sigma'}\left[i\frac{\partial}{\partial x^\mu}\delta_i{}^j + gA^a_\mu\left(\hat{T}^a\right)_i{}^j\right] - m\,\delta_\sigma{}^{\sigma'}\delta_i{}^j\right\}\psi_{\sigma',j}. \tag{8B.1.1}$$

  Quark field $\psi \to \psi_{\sigma',j}$ carries two indices:

  1. Spinor index σ

     $\sigma \in \{1,2,3,4\}$ SO(3,1) $\sim$ SO(4) Lorentz group; σ is a direct sum of left- and right- handed Weyl components

     By the way,
     $$\hat{\gamma}^\mu\,\hat{\gamma}^\nu + \hat{\gamma}^\nu\,\hat{\gamma}^\mu = -2\eta^{\mu\nu}\hat{1}_\sigma$$
     is a (non-compact) **Clifford algebra**. The Euclidean spacetime versions offer a natural basis of operators acting on the spinor representation(s) of SO(5) [or SO(4)]. Module 10.
  2. Color index i
     $$i \in \{1, 2, \ldots, n\} \qquad \text{defining representation of (e.g.) su}(n).$$

  – With $\hat{A}_\mu = 0$, theory is invariant under global (indept. of spacetime) SU(n) transformations on color. Also invariant under Lorentz transforms (which however act on both spinor indices and spacetime arguments $\Rightarrow$ relativistic spin-orbit coupling).

- The fermion field ψ (Dirac adjoint field $\bar{\psi}$) transforms in the defining "n" (conjugate "$\bar{n}$") representation of the color symmetry group, e.g. su(n).
- The gauge field $\hat{A}_\mu = A^a_\mu \hat{T}^a$ transforms in the adjoint representation under **global** color symmetry transformations. (It also transforms like a vector under Lorentz transformations).
- Local spacetime color symmetry:

$$\begin{aligned} \psi(x) &\to \hat{U}(x)\,\psi(x), \\ \bar{\psi}(x) &\to \bar{\psi}(x)\,\hat{U}^\dagger(x), \\ \hat{A}_\mu(x) &\to \hat{U}(x)\,\hat{A}_\mu(x)\,\hat{U}^\dagger(x) + \frac{i}{g}\hat{U}(x)\frac{\partial}{\partial x^\mu}\hat{U}^\dagger(x) \end{aligned} \tag{8B.1.2}$$

For a local symmmetry transformation, $\hat{A}_\mu(x)$ does **not** transform in the adjoint rep. There is a correction.

QCD of quarks and gluons: color group is SU(3). 3 independent types of each quark. (That means 3 different colors of "up" quarks. There are also 5 other flavors of quarks; the 6 flavors divide into 3 generations. Flavors have nothing to do with color, however, and can be thought of as yet another index.) There are 8 independent types of gluons, corresponding to states of the adjoint representation of the color group.

Thus we get physical implementations of the defining (1,0), conjugate (0,1) and adjoint (1,1) representations of su(3). Dynamics for the gauge field: additional term in Lagrangian involving the (non-abelian) field strength tensor

$$\hat{F} = d\hat{A} + g\hat{A} \wedge \hat{A}.$$

N.B., can also get classical general relativity from very similar construction. **Gauge the Lorentz group SO(3,1)!**.

- Gauge fields: Christoffel symbols or spin connections.
- Field strength tensor: Riemann tensor.
- Problem: quantum fluctuations become too strong on short scales; "straightforward" quantization of GR gives a nonrenormalizable QFT that fails to make predictions in the ultraviolet (highest energy, shortest distance) limit. Not completely unexpected, since one might need to deal with quantum fluctuations of spacetime itself, and that isn't captured by a perturbative quantization of GR. Way out? Strings (but...no SUSY evidence; landscape problem)? Loop quantum gravity? Something else?

8B.2 Generalized quantum magnets

- In condensed matter, frequently want to understand the physics of quantum Heisenberg antiferromagnets with spin SU(2) symmetry. E.g.,

$$H = J_1 \sum_{\langle \mathbf{r},\mathbf{r}' \rangle} \hat{S}^a_{\mathbf{r}} \hat{S}^a_{\mathbf{r}'} + J_2 \sum_{\langle\langle \mathbf{r},\mathbf{r}' \rangle\rangle} \hat{S}^a_{\mathbf{r}} \hat{S}^a_{\mathbf{r}'} + \ldots, \tag{8B.2.1}$$

where (e.g.) $2\hat{S}^a_{\mathbf{r}}$ is a 2×2 Pauli matrix acting on a spin-1/2 degree of freedom sitting on site $\mathbf{r}$ in some lattice. Coupling ("exchange") constants J_1, J_2, etc. encode interactions: nearest-neighbor exchange, next-nearest-neighbor, etc.

- Quantum magnets with antiferromagnetic exchange ($J_1 > 0$) can have complicated ground states, e.g. Néel state on a bipartite lattice, root-3 state on triangular lattice, columnar states, valence bond solids, spin liquids, etc. Frustration and quantum fluctuations mean that the ground state is never a simple product state (unlike in a simple quantum ferromagnet).
- In low dimensions, quantum fluctuations are strong and Hamiltonian such as Eq. (8B.2.1) can be strongly coupled. Perturbative calculations fail. **What to do?**
- No small parameter: **invent one.** For example, can replace SU(2) $\Rightarrow$ SU(n); hope to use $1/n$ as small parameter.
- Problem: for spin-1/2 SU(2) magnet, can always form a singlet from two nearest-neighbor sites

$$|0\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right].$$

But if we put a defining "n" representation "spin" on each site, we know that we need n sites to form a singlet in su(n). Either give up notion of local singlets, or modify construction. Local singlets are good if we want to find valence bond solids or spin liquids...

- One way: put a defining representation ψ_i on half of the sites ("A" sites on bipartite lattice) and a conjugate representation ϕ^i on the other half ("B" sites on a bipartite lattice). Then we can form a scalar by pairing one A and one B site: $\phi^i \psi_i$.

- How to make this concrete? What is the Hilbert space on each site, and how does the Hamiltonian act on it? Answer: **(E.g.) slave fermions**. The generalization of Eq. (8B.2.1) to SU(n) is

$$H = J_1 \sum_{\langle \mathbf{r},\mathbf{r'} \rangle} \hat{T}^a_{\mathbf{r}} \hat{T}^a_{\mathbf{r'}} + \ldots , \tag{8B.2.2}$$

where

$$\hat{T}^a = c^{\dagger\, i} \left[\hat{t}^a \right]_i^{\ j} c_j , \tag{8B.2.3}$$

and c_j is a fermion operator that transforms in the defining n (ω_1) representation. $\left[\hat{t}^a \right]_i^{\ j}$ is the explicit matrix generator in this representation. Can easily verify that

$$[\hat{T}^a, \hat{T}^b] = i f^{abc} \hat{T}^c \tag{8B.2.4}$$

if $[\hat{t}^a, \hat{t}^b] = i f^{abc} \hat{t}^c$ is the su(n) Lie algebra commutation relations.

- That gives us the operators in the Hamiltonian. What about the states?

$$\text{Defining:} \quad \psi_i \Leftrightarrow c^\dagger_i \left| 0 \right\rangle \qquad \text{set of all states with one and only one particle per site} \tag{8B.2.5}$$

$$\text{Conjugate:} \quad \phi^i \Leftrightarrow c^\dagger_{i_1} c^\dagger_{i_2} \times \cdots \times c^\dagger_{i_{n-1}} \left| 0 \right\rangle \qquad \text{set of all states with } n-1 \text{ particles per site} \tag{8B.2.6}$$

We can then form a singlet

$$\epsilon^{i_1 \cdots i_n} \psi_{i_1} \phi_{i_2 \cdots i_n} .$$

- The above treats different sites asymmetrically. What if we want all sites to have the same degrees of freedom? Take $n = 2m$, and assign $\psi_{[i_1 \cdots i_m]} \Leftrightarrow \omega_m$ representation to each site. The singlet is

$$\epsilon^{i_1 \cdots i_m i_{m+1} \cdots i_n} \psi_{[i_1 \cdots i_m]}(\mathbf{r})\, \psi_{[i_{m+1} \cdots i_{2m}]}(\mathbf{r'})$$

- Using slave fermions in the defining representation, Eq. (8B.2.3) still holds for the Hamiltonian. The states on each site have m fermions,

$$\psi_{[i_1 \cdots i_m]} \Leftrightarrow c^\dagger_{i_1} c^\dagger_{i_2} \times \cdots \times c^\dagger_{i_m} \left| 0 \right\rangle . \tag{8B.2.7}$$

Other schemes:

- Slave bosons. Natural Fock space states are completely symmetric tensors (representations with weights $m\omega_1$).

- More complicated slave fermions–add an additional "flavor" index (or more). Can realize generic tensor representation of SU(n) on each site.

- Use a different group. E.g. Sp($2n$). Then we can form a singlet from two defining representation indices, using the antisymmetric "metric" ϵ^{ij}.

8B.3 Non-linear sigma models

8B.4 Kac-Moody 2D conformal field theories