8. Highest-weight representations

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The discussion here follows chapter X of [1], chapter 13 of [2], and elements of Chapters 10–13 in [3].

8.1 Weights in 3 bases

In this module we focus on generic irreducible highest-weight representations, wherein we assume the existence of some state that cannot be raised using simple root vector operators, and we use the Master Weight Depth Formula (MWDF) Eq. (5.1.23) to obtain all other weights below it in the representation.
The emphasis so far has been on expressing weights in terms of the $n$ simple roots $\{\alpha_i\}$ for a rank-$n$ Lie algebra. In the adjoint representation, the set of positive roots consists of simple root combinations with non-negative integral coefficients. For a general representation, however, the coefficients are only required to be rational numbers. In order to specify a generic highest-weight representation, it is advantageous to develop a basis for the space of weights $H^*$ in which expansion coefficients are integral.

### 8.1.1 Roots, coroots, and fundamental weights

In fact, let us specify three different bases, each of which gives certain advantages. Let $\Lambda$ be a weight in some irreducible representation. Then

<table>
<thead>
<tr>
<th>Bases for a generic weight $\Lambda \in H^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda = \sum_{j=1}^{n} \kappa_j^{(\Lambda)} \alpha_j$, Expansion in terms of simple roots,</td>
</tr>
<tr>
<td>$= \sum_{j=1}^{n} \kappa_j^{(\Lambda)} \alpha_j'$, Expansion in terms of simple coroots,</td>
</tr>
<tr>
<td>$= \sum_{j=1}^{n} \Lambda_j \omega_j$, Expansion in terms of fundamental weights.</td>
</tr>
</tbody>
</table>

As defined in Eq. (5.1.23), the coroot $\alpha_i'$ associated to a root $\alpha_i$ is defined via

$$\alpha_i' = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}.$$  \hspace{1cm} (8.1.2)

The Cartan matrix is expressed as the scalar product between roots and coroots, $A_{ij} = \langle \alpha_i, \alpha_j' \rangle$ [Eq. (5.3.8)]. The expansion coefficients $\{\kappa_i^{(\Lambda)}, \kappa_j^{(\Lambda)}\} \in \mathbb{Q}$ (the field of rational numbers) [Prop. VIII., Sec. 5.2.2].

The fundamental weights $\{\omega_i\}$ ($1 \leq i \leq n$) are defined to be dual to the set of simple coroots, i.e. these are mutually orthonormal:

$$\langle \omega_i, \alpha_j' \rangle = \delta_{ij}.$$  \hspace{1cm} (8.1.3)

Since the simple coroots span $H^*$, the fundamental weights do as well. The coefficients $\{\Lambda_j\}$ for the expansion of a weight $\Lambda$ in terms of fundamental weights are the Dynkin labels, which are guaranteed to be integers. This follows from taking the inner product of the expansion with a simple coroot,

$$\langle \Lambda, \alpha_i' \rangle = \Lambda_i = (m-p)\Lambda_{i||},$$  \hspace{1cm} (8.1.4)

where $m$ and $p$ respectively denote the number of weights below and above $\Lambda$ in the chain generated by acting with $E_{i||}$. [Eq. (5.1.23)].

We then immediately see that a highest weight $\Lambda$ is defined by a set of non-negative Dynkin coefficients $\{\Lambda_i \geq 0\}$, since by definition $p_{\Lambda,\pi_i} = 0 \forall \pi_i$. Thus we can encode any representation by the $n$ non-negative Dynkin coefficients of its highest weight. On the other hand, the expansion of weights in terms of simple roots is also useful, since it encodes the geometric picture for the representation connected to root scalar products. To relate these, we write the overlap matrices

$$\langle \pi_i, \pi_j \rangle = A_{ij} \frac{||\pi_i||^2}{2},$$  \hspace{1cm} (8.1.5a)

$$\langle \alpha_i', \alpha_j' \rangle = \frac{2}{||\alpha_i||^2} A_{ij},$$  \hspace{1cm} (8.1.5b)

$$\langle \omega_i, \omega_j \rangle = F_{ij}.$$  \hspace{1cm} (8.1.5c)

In these equations $A_{ij}$ denotes the Cartan matrix. The symmetric overlap matrix between fundamental weights $F_{ij}$ is called the quadratic form matrix. Its relation to the Cartan matrix follows from

$$\Lambda_j = \langle \Lambda, \alpha_i' \rangle = \sum_{i=1}^{n} \kappa_i^{(\Lambda)} A_{ij},$$

$$\kappa_j^{(\Lambda)} = \langle \Lambda, \omega_j \rangle = \sum_{i=1}^{n} \Lambda_i F_{ij},$$

$$\Rightarrow \Lambda_j = \sum_{i=1}^{n} \kappa_i^{(\Lambda)} \frac{2}{||\pi_i||^2} A_{ij} = \sum_{i=1}^{n} \Lambda_i \sum_{i=1}^{n} F_{ij} \frac{2}{||\pi_i||^2} A_{ij} \Rightarrow \sum_{i=1}^{n} F_{ij} \frac{2}{||\pi_i||^2} A_{ij} = \delta_{ij},$$  \hspace{1cm} (8.1.6)
so that
\[(\hat{F}^{-1})_{ij} = \frac{2}{|\alpha_i|^2} A_{ij} = (\overline{\alpha}_i^\dagger, \overline{\alpha}_j^\dagger).\] (8.1.7)

The quadratic form matrices for all 9 classes of semi-simple Lie algebras are known explicitly. These are tabulated for the four classical Lie algebra families in Sec. 8.6; the exceptional cases can be found in section 13.A of [2].

For the moment, we note that Eq. (8.1.6) implies that
\[\Lambda = \kappa^{(A)} \hat{A}, \quad \kappa^{(A)} = \Lambda \hat{A}^{-1},\] (8.1.8)
where we view $\Lambda$ and $\kappa^{(A)}$ as row vectors of coefficients. Consider the defining or $\mathbf{3}$ representation of $\text{su}(3)$, previously analyzed in Sec. 5.1. In Eq. (5.1.8), we found that the weights in this representation can be expressed as
\[
M^A = \frac{2}{3} \overline{\alpha}_1 + \frac{1}{3} \overline{\alpha}_2,
\]
\[
M^B = -\frac{1}{3} \overline{\alpha}_1 + \frac{1}{3} \overline{\alpha}_2,
\]
\[
M^C = -\frac{1}{3} \overline{\alpha}_1 - \frac{2}{3} \overline{\alpha}_2,
\] (8.1.9)
where $\alpha_+ \rightarrow \overline{\alpha}_1$ and $\alpha_u \rightarrow \overline{\alpha}_2$. Using the $\text{su}(3)$ Cartan matrix from Eq. (5.3.15), the associated Dynkin coefficients are [via Eq. (8.1.8)]
\[
M^A \rightarrow (1, 0),
\]
\[
M^B \rightarrow (-1, 1),
\]
\[
M^C \rightarrow (0, -1).
\] (8.1.10)

We can obtain these another way, using the MWDF Eq. (5.1.23). This is the complement to the positive-root-building algorithm we used in Secs. 5.3.2 and 7.1.1. Now, we subtract simple roots to generate all of the weights from the highest one:

1. We start with the highest weight state
\[
\begin{bmatrix}
1 \\
0
\end{bmatrix}.
\] (8.1.11a)

The depths are given by the Dynkin coefficients,
\[
m_{A, \overline{\alpha}_1} = 1, \quad m_{A, \overline{\alpha}_2} = 0.
\] (8.1.11b)

2. At the second depth level, we know that $\Lambda^{(2)} \equiv \Lambda - \overline{\alpha}_1$ is a valid weight. Its Dynkin coefficients are
\[
\begin{bmatrix}
-1 \\
1
\end{bmatrix}.
\] (8.1.11c)

![Figure 8.1: Weight trees for the defining “3” (1,0), conjugate “\(\overline{3}\)” (0,1), and adjoint (1,1) representations of su(3). The ordered pair labels ($\Lambda_1, \Lambda_2$) ($\Lambda_i \in \mathbb{Z}$) are Dynkin coefficients. The simple root labels {$\overline{\alpha}_i$} indicate the difference between the weights above and below the connecting line.](image-url)
Figure 8.2: Weight geometries for the defining \( ^3 \) (1,0), conjugate \( ^{-3} \) (0,1), and adjoint (1,1) representations of \( \text{su}(3) \). The 3 and \( ^{-3} \) representations are also faintly superimposed over the adjoint. These three representations belong to the three different conjugacy classes (shifted triangular lattices) of \( \text{su}(3) \) weights.

The heights are

\[
p_{\Lambda^{(2)}; \vec{\alpha}_1} = 1, \quad p_{\Lambda^{(2)}; \vec{\alpha}_2} = 0, \tag{8.1.11d}
\]

so that the depths below this level are

\[
m_{\Lambda^{(2)}; \vec{\alpha}_1} = p_{\Lambda^{(2)}; \vec{\alpha}_1} + \Lambda^{(2)}_1 = 1 - 1 = 0, \tag{8.1.11e}
\]

\[
m_{\Lambda^{(2)}; \vec{\alpha}_2} = p_{\Lambda^{(2)}; \vec{\alpha}_2} + \Lambda^{(2)}_2 = 0 + 1 = 1.
\]

3. At the third depth level, we know that \( \Lambda^{(3)} = \Lambda^{(2)} - \vec{\alpha}_2 \) is a valid weight. Its Dynkin coefficients are

\[
\begin{bmatrix}
0 \\
-1
\end{bmatrix}.
\]

The heights are

\[
p_{\Lambda^{(3)}; \vec{\alpha}_1} = 0, \quad p_{\Lambda^{(3)}; \vec{\alpha}_2} = 1, \tag{8.1.11f}
\]

so that the depths below this level are

\[
m_{\Lambda^{(3)}; \vec{\alpha}_1} = p_{\Lambda^{(3)}; \vec{\alpha}_1} + 0 = 0, \quad m_{\Lambda^{(3)}; \vec{\alpha}_2} = p_{\Lambda^{(3)}; \vec{\alpha}_2} - 1 = 0. \tag{8.1.11g}
\]

We conclude that \( \Lambda^{(3)} \) is the lowest weight state.

In addition to the defining representation (1,0) \( \leftrightarrow \omega_1 \), the \( A_2 = \text{su}(3) \) algebra has a second fundamental representation (0,1) \( \leftrightarrow \omega_2 \). This turns out to be the conjugate or \( ^{-3} \) representation. The weight trees for the 3, \( ^{-3} \), and adjoint representations of \( \text{su}(3) \) are depicted in Fig. 8.1. We can use Eq. (8.1.8) to obtain the expansion coefficients of all of these weights in terms of simple roots; the simple root geometry then implies the geometry of the representations shown in Fig. 8.2. Note that these three representations (1,0), (0,1), and (1,1) correspond to regular fragments of three different triangular lattices. The lattices are generated by the same lattice vectors \( \vec{\alpha}_1 \) and \( \vec{\alpha}_2 \), but from different (shifted) origins. The different lattices are the three conjugacy classes of \( \text{su}(3) \), exemplified by these representations. All other representations reside on one of these lattices.

Conjugate representations will be defined precisely below, but the key features appear in Figs. 8.1 and 8.2. Namely, the top weight of the conjugate \( ^{-3} \) representation is minus the bottom weight of the defining 3 representation (and vice-versa). All weights are in one-to-one correspondence between a representation and its conjugate, being related by inversion (\( \Lambda \rightarrow -\Lambda \)). Finally, the adjoint representation is self-conjugate, since its lowest weight (-1,-1) is minus its highest one.

8.1.2 Marks and comarks

The highest weight state of the adjoint representation for any rank-\( n \) Lie algebra is the highest root \( \theta \), introduced in Sec. 5.3.1. It can be expanded as in Eq. (8.1.1),

\[
\theta = \sum_{j=1}^{n} a_j \overline{\omega}_j = \sum_{j=1}^{n} a_j' \overline{\omega}'_j = \sum_{j=1}^{n} \theta_j \omega_j. \tag{8.1.12}
\]
Figure 8.3: Dynkin diagrams for the four classical families $A_n = \text{su}(n + 1)$, $B_n = \text{so}(2n + 1)$, $C_n = \text{sp}(2n)$, and $D_n = \text{so}(2n)$. The three numbers $(m_1, m_2, m_3)$ that label each node specify the fundamental weight $\omega_{m_1}$, the mark for the associated root $a_{m_2}$, and the comark for the associated coroot $a_{m_3}^\vee$. Here we set $|\alpha|^2 = 2$, where $\alpha$ is a long root, so that marks and comarks are identical for simply-laced algebras.

In this case, the root and coroot coefficients are called marks $\{a_j\}$ and comarks $\{a_j^\vee\}$, respectively. The comarks play a key role in the construction of dominant highest-weight representations in affine Lie algebras, which are equivalent to certain exactly solvable 2D conformal field theories (Kac-Moody current algebras). The marks and comarks can be computed from the Dynkin coefficients of $\theta$ via Eq. (8.1.8). For $\text{su}(3)$, $\theta = (1, 1)$. We will derive the $\{\theta_j\}$ for the four classical families in Sec. 8.2.4, below.

Another key piece of data for any Lie algebra is the dual Coxeter number $g$, defined via

$$g = 1 + \sum_{j=1}^n a_j^\vee. \tag{8.1.13}$$

In module 9, we will show that $g$ is half the value of the quadratic Casimir operator, evaluated in the adjoint representation. The dual Coxeter numbers for the four classical families are given in Sec. 8.6.

We can use Dynkin diagrams to enumerate the fundamental weights for each rank-$n$ algebra. These are shown in Fig. 8.3, along with the corresponding marks and comarks. Here and in Sec. 8.6, we use a normalization scheme such that long roots have $|\alpha|^2 = 2$.

### 8.2 Mostly su($n + 1$)

#### 8.2.1 Elementary tensors representations: complete (anti)symmetrization

We will build up more complex representations from simpler ones by tensoring together the latter. If we have two representations $\phi_i$ and $\eta_j$ such that for $x \in L$,

$$\begin{align*}
(\hat{X}^{(1)} \phi)_i &= X^{(1)}_{ij} \phi_j, \\
(\hat{X}^{(2)} \eta)_i &= X^{(2)}_{ij} \eta_j,
\end{align*} \tag{8.2.1}$$

then the action on the tensor (direct) product is

$$\left(\hat{X} \phi \otimes \eta\right)_i = X^{(1)}_{ik} \phi_k \eta_j + X^{(2)}_{jk} \phi_i \eta_k. \tag{8.2.2}$$

Thus if $\phi$ and $\eta$ are weight vectors with weights $M^{(1)}$ and $M^{(2)}$, respectively, then $\phi \otimes \eta$ is also a weight vector with weight $M^{(1)} + M^{(2)}$. In particular, if we tensor together two highest weight representations with highest weights $\Lambda^{(1)}$ and $\Lambda^{(2)}$, respectively, then the highest weight of the tensor product is given by $\Lambda^{(1)} + \Lambda^{(2)}$.

In what follows, we will exploit the following fact, which we do not prove here:
• Claim: The weight space\(^1\) (degeneracy) of the highest weight is always one-dimensional for an irreducible representation.

If we tensor together \(m\) different copies of the defining representation in some Lie algebra (i.e. form an \(m\)-fold direct product), the resulting rank-\(m\) tensor \(T_{i_1i_2\cdots i_m}\) will \textit{not} belong to an irreducible representation. As discussed in the previous section, any highest-weight representation in a rank-\(n\) Lie algebra can be specified by the \(n\) non-negative Dynkin labels of its highest weight \(\Lambda = (\Lambda_1,\Lambda_2,\ldots,\Lambda_n)\). We can in fact generate most or all\(^2\) representations by appropriately symmetrizing and antisymmetrizing groups of indices of the unsymmetrized \(T_{i_1i_2\cdots i_m}\). The simplest cases correspond to complete antisymmetrization or complete symmetrization. We now show how to obtain the Dynkin coefficients for \(\text{su}(n+1)\) tensors in these cases.

### 8.2.1.1 Complete antisymmetrization: the fundamental weight representations

Consider the defining \("n+1"\) representation of \(A_n = \text{su}(n+1)\). As in \(\text{su}(3)\) (Figs 8.1 and 8.2), the Dynkin labels for the \(n+1\) representation are \((\Lambda_1,\Lambda_2,\Lambda_3,\ldots,\Lambda_n) = (1,0,0,\ldots,0)\). Equivalently, we say that the defining representation is equal to the first fundamental weight,

\[
\Lambda = \omega_1.
\]

Using the \(n \times n\) Cartan matrix for \(A_n\) [Eq. (7.1.23)]

\[
\hat{A}_\Lambda = \begin{bmatrix}
2 & -1 & 0 & \ldots & 0 & 0 \\
-1 & 2 & -1 & \ldots & 0 & 0 \\
0 & -1 & 2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 2 & -1 \\
0 & 0 & 0 & \ldots & -1 & 2 \\
\end{bmatrix},
\]

we can generate the \(n\) descendants of the highest weight using the MWDF algorithm in Eq. (8.1.11). This gives the defining representation weight tree shown in Fig. 8.4. Note that the \(m^{th}\) step down the chain is generated by lowering with the simple root \(\alpha_m\) (\(1 \leq m \leq n\)). This is also easily seen from the explicit matrix representation of the simple root vectors \(\{E_{\alpha_m}\}\) for \(A_n\), Eqs. (7.1.1) and (7.1.9). Since there are \(n + 1\) different weights in Fig. 8.4, all weight spaces in the defining representation are one-dimensional (non-degenerate).

We can view the \(n+1\) representation as a rank-1 tensor \(T_i, i \in \{1, 2, \ldots, n+1\}\). Now, consider the completely antisymmetrized rank-\(m\) tensor

\[
T_{[i_1i_2\cdots i_m]}, \quad 1 \leq m \leq n+1.
\]

This has

\[
\frac{(n+1)!}{m!(n+1-m)!}
\]

independent components. We assert that this \(m\)-fold tensor is automatically an irreducible representation. The argument is twofold:

1. We cannot simplify the index structure further, i.e. symmetrize any subgroup of indices,
2. We cannot contract any two indices to form a simpler tensor.

The second fact is equivalent to the statement that there is no natural “metric” for \(\text{su}(n+1)\) \((n \geq 2)\) that can be used to contract pairs of defining representation indices to produce an invariant scalar, unlike in \(\text{sp}(2n)\) or \(\text{so}(n)\) (module 2A). [We will discuss contractions using the rank-\((n+1)\) Levi-Civita symbol when we introduce conjugate representations, below.]

What are the Dynkin coefficients for \(T_{[i_1i_2\cdots i_m]}\)?

- \(m = 2\): second-rank, antisymmetric tensor \(T_{ij}\) with \((n+1)n/2\) components. The highest weight possible for each index \(i\) and \(j\) is \(\omega_1 = (1,0,0,\ldots,0)\), associated to \(i = 1\) or \(j = 1\). However, due to the antisymmetrization, we cannot have \(i = j\). Thus, the component with the highest possible weight is \(T_{12} = -T_{21}\). From Fig. 8.4, this has weight

\[
(1,0,0,\ldots,0) + (-1,1,0,\ldots,0) = (0,1,0,\ldots,0) \rightarrow \omega_2.
\]

Thus the case \(m = 2\) gives a realization of the representation with highest weight \(\Lambda = \omega_2\).

---

\(^1\)Recall from module 5 that the weight space is the degeneracy of a particular weight \(\Lambda\) in some irreducible representation. If we consider all possible representations in the same conjugacy class, then the weight space must be specified for all “sites” of the corresponding infinite weight lattice. Each weight space is therefore characterized by a non-negative integer. This is not to be confused with \(H^*_+\), which is the space of weights, i.e. the space in which different weights are expressed as linear combinations of simple roots, simple coroots, or fundamental weights. The terminology here is a bit unfortunate.

\(^2\)All representations in \(A_n = \text{su}(n+1)\) and \(C_n = \text{sp}(2n)\) can be obtained by tensoring together copies of the defining representation, and (anti)symmetrizing. In \(B_n = \text{so}(2n+1)\) and \(D_n = \text{so}(2n)\), however, there are spinor representations that cannot be obtained by tensoring together vector indices. We will demonstrate this in Sec. 8.3.
Figure 8.4: Weight tree for the defining representation of su$(n+1)$ ($A_n$): $(1, 0, 0, \cdots, 0) \leftrightarrow \omega_1$. The simple root labels $\{\pi_i\}$ indicate the difference between the weights above and below the connecting line.

- $m = 3$: $T_{[ijk]}$ with $(n + 1)n(n - 1)/3!$ components. The component with the highest possible weight is $T_{123} = -T_{213} = T_{231}$ (etc.). The highest weight is
  \[
  (1, 0, 0, 0, \ldots, 0) + (-1, 1, 0, 0, \ldots, 0) + (0, -1, 1, 0, \ldots, 0) = (0, 0, 1, \ldots, 0) \rightarrow \omega_3. \tag{8.2.6b}
  \]
  
- $m = n$: $T_{[i_1i_2\cdots i_n]}$ with $n + 1$ components. The highest weight is
  \[
  (1, 0, 0, \ldots, 0, 0, 0) + (-1, 1, 0, 0, \ldots, 0, 0) + \ldots + (0, 0, 0, \ldots, 0, -1, 1) = (0, 0, 0, \ldots, 0, 0, 1) \rightarrow \omega_n. \tag{8.2.6c}
  \]
- $m = n + 1$: $T_{[i_1i_2\cdots i_{n+1}]}$. This is proportional to the rank-$(n+1)$ Levi-Civita symbol $\epsilon_{i_1\cdots i_{n+1}}$ and has only one unique component (which may be zero); see also Sec. 8.2.3, below. The weight is
  \[
  (1, 0, 0, \ldots, 0, 0, 0) + (-1, 1, 0, 0, \ldots, 0, 0) + \ldots + (0, 0, 0, \ldots, 0, 0, -1) = (0, 0, 0, \ldots, 0, 0, 0) = 0. \tag{8.2.6d}
  \]

- Thus for su$(n+1)$, the fundamental weight $\omega_m$ corresponds precisely to a rank-$m$ completely antisymmetric tensor:
  \[
  T_{[i_1i_2\cdots i_m]}, \quad 1 \leq m \leq n: \text{irreducible representation of su}(n+1) \text{ with highest weight } \omega_m. \tag{8.2.7}
  \]

The same conclusion will apply to all but one (two) fundamental weights in $B_n = \text{so}(2n+1)$ [$D_n = \text{so}(2n)$]; the outliers are the fundamental spinor representations. In $C_n = \text{sp}(2n)$, the fundamental weight $\omega_m$ also corresponds to a rank-$m$ completely antisymmetric tensor $\tilde{T}_{[i_1\cdots i_m]}$ ($1 \leq m \leq n$); the tilde indicates that, in addition, all traces with the antisymmetric “symplectic metric” $\epsilon^{ij}$ have been removed [c.f. Eqs. (7.2.1) and (7.2.2)].

### 8.2.1.2 Complete symmetrization

Another class of representations are the completely symmetric rank-$m$ tensors,

\[
T_{(i_1i_2\cdots i_m)}, \quad m \geq 1. \tag{8.2.8}
\]

This has

\[
\frac{(m+n)!}{m!n!} \tag{8.2.9}
\]

independent components. For su$(n+1)$, completely symmetric tensors are also irreducible representations; the argument is the same as the antisymmetric case. The highest weight state of the rank-$m$ tensor is $T_{11\cdots 1}$, with weight $\Lambda = (m, 0, 0, \cdots, 0)$:

\[
T_{(i_1i_2\cdots i_m)}, \quad m \geq 1: \text{irreducible representation of su}(n+1) \text{ with highest weight } m \omega_1. \tag{8.2.10}
\]

A completely symmetric rank-$m$ tensor forms an irreducible representation with weight $m \omega_1$ for $C_n$ as well. For the orthogonal algebras one has to in addition remove all possible traces to get irreducible representations.

Thus we have identified tensor representations with highest weight $\Lambda$ given by the fundamental weights $\Lambda \in \{\omega_i\}$ (completely antisymmetric) or by $\Lambda = m \omega_1$ (completely symmetric). Other irreducible representations of su$(n+1)$ can be built from tensors with mixed indices; one technique (Young tableaux) will be demonstrated below in Sec. 8.5.
Figure 8.5: Symmetric second rank tensor \((2,0) \Leftrightarrow T_{(ij)}\) representation of \(su(3)\). This is in the conjugacy class of the conjugate \(\bar{3} \Leftrightarrow (0,1)\) representation. There are \(3(4)/2 = 6\) states in this representation, so all weight spaces are one dimensional.

### 8.2.2 \(su(3)\) again

We next construct the weight trees and geometries for a couple of more complex \(su(3)\) representations. The fundamental representations \(\omega_1\) and \(\omega_2\) were depicted in Figs. 8.1 and 8.2, along with the adjoint representation \(\omega_1 + \omega_2 \rightarrow (1,1)\). The second-rank symmetric tensor representation \(T_{(ij)} \Leftrightarrow (2,0)\) is shown in Fig. 8.5. A mixed tensor representation with Dynkin labels \((2,1)\) is shown in Fig. 8.6. This representation turns out to have 15 total states, but only 12 distinct weights. In this case, each of the “inner triangle” weights is doubly degenerate, as we will prove later in module 9.

Figure 8.6: \((2,1)\) representation of \(su(3)\). This is in the conjugacy class of the defining \(3 \Leftrightarrow (1,0)\) representation. There are 15 states in this representation, but only 12 distinct weights. Each of the weights in the inner triangle are doubly degenerate, as we will show using Freudenthal’s formula in module 9.

### 8.2.3 The Levi-Civita symbol, \(n+1\) representation of \(su(n+1)\), and conjugate representations

We argued above that completely (anti)symmetric tensors are automatic irreducible representations of \(su(n+1)\). Part of the argument was that there is no natural “metric” with which we can “tie up” pairs of defining representation indices to form a “singlet” (group invariant) in \(A_n\). In fact, there is one very important tensor that is a group invariant: the rank-\((n+1)\) completely antisymmetric tensor (Levi-Civita symbol\(^3\)):

\[
\epsilon^{i_1i_2\ldots i_{n+1}} = \epsilon^{[i_1i_2\ldots i_{n+1}]}, \quad \epsilon^{1,2,3,\ldots n,n+1} = +1. \tag{8.2.11}
\]

Here we write this object with all indices “upstairs,” as opposed to the tensor in Eq. (8.2.4) built out of defining, “downstairs” \(\omega_1\)-indices. Following conventions from general relativity and differential geometry, the idea is that we can only “contract” (sum

---

\(^3\)Here we use the terms “Levi-Civita symbol” and “Levi-Civita tensor” interchangeably. There is a distinction for coordinate tensors on curved manifolds, but our indices here are constructed from the defining representation in a linear vector space. This is independent of the arbitrary coordinates used to parameterize the corresponding Lie group manifold.
pairwise) over one upper and one lower index, but never pairs of lower or upper indices. If we have a rank \( m \geq n + 1 \) tensor, then we can contract
\[
e^{i_1 j_2 \cdots i_{n+1}} T_{i_1 j_2 \cdots i_{n+1} i_{n+2} \cdots i_m} = \epsilon^{i_1 j_2 \cdots i_{n+1}} T_{[i_1 \cdots i_{n+1}] j_{n+2} \cdots i_m} \equiv \tilde{T}_{i_{n+2} j_{n+3} \cdots i_m}, \quad m \geq n + 1. \tag{8.2.12}
\]

Here repeated indices are summed (Einstein). The square brackets on the right-hand side of the first line mean complete antisymmetrization: the Levi-Civita symbol projects out this combination of components. In particular, if \( m = n + 1 \), then \( \tilde{T} \) (if non-zero) is guaranteed to be a scalar, i.e. not to transform under SU\((n + 1)\) transformations. The reason is simply because the Levi-Civita symbol has only one unique component. Thus \((n + 1)\) different \( \omega_1 \)-indices are needed to form a “singlet” in su\((n + 1)\).

We can also use the Levi-Civita symbol to “raise” indices on a completely antisymmetric rank-\( m \leq n \) tensor:
\[
e^{i_1 j_2 \cdots i_{n+1} i_{m+1} \cdots i_{n+1}} T_{[i_1 \cdots i_{m}]} \equiv T^{[i_1 \cdots i_{m+1} \cdots i_{n+1}]}, \quad m \leq n. \tag{8.2.13}
\]

This gives an alternative tensor representation for the \( \omega_m \) representation as an antisymmetric tensor with \((n + 1 - m)\) indices. In particular, the conjugate \( \bar{n} + 1 \) representation is defined via
\[
\tilde{T}^i \equiv \epsilon^{i_1 j_2 \cdots j_n} T_{[i_1 j_2 \cdots j_n]}, \quad \bar{n} + 1 = \omega_n \text{ conjugate representation of } \text{su}(n + 1). \tag{8.2.14}
\]

The \( T^{[i_1 \cdots i_m]} \) can be viewed as the completely antisymmetrized tensor of \( m \) \((\bar{n} + 1)\)-representation indices; this is itself an irreducible representation with highest weight \( \Lambda = \omega_{n+1-m} \) \((1 \leq m \leq n)\). The defining \( n + 1 \) representation is therefore equivalent to the \( n \)-th rank tensor \( T^{[i_1 \cdots i_n]} \). The precise relation obtains by introducing the Levi-Civita symbol with all lower indices,
\[
\epsilon_{i_1 j_2 \cdots i_{n+1}} = \epsilon_{[i_1 j_2 \cdots i_{n+1}]}; \quad \epsilon_{1,2,3,\ldots,n,n+1} = +1, \tag{8.2.15}
\]
so that
\[
T_i = \frac{1}{n!} \epsilon_{i j_1 \cdots j_n} T^{[j_1 j_2 \cdots j_n]} = \frac{1}{n!} \epsilon_{i j_1 \cdots j_n} \epsilon^{k j_1 \cdots j_n} T_k = T_i. \tag{8.2.16}
\]

We can also consider tensors with mixed upper and lower indices. The simplest is the Kronecker delta \( \delta^i_j \), which is automatically an invariant tensor. A general second-rank tensor \( T^i_j \) instead decomposes into
\[
\text{scalar (0)}: \quad T_0 \equiv T^i_i, \tag{8.2.17a}
\]
\[
\text{adjoint (1,1)}: \quad \tilde{T}^i_j \equiv T^i_j = \frac{\delta^i_j}{n!} T^k_j. \tag{8.2.17b}
\]
Thus the pairwise contraction of defining \( n + 1 \) and conjugate \( \bar{n} + 1 \) indices gives a scalar, but this can be alternatively viewed as the contraction with the \( n + 1 \)-fold Levi-Civita tensor:
\[
U^i V_i = \epsilon^{i_1 j_2 \cdots i_{n+1}} U_{[i_1 \cdots i_n]} V_{i_{n+1}}. \tag{8.2.18}
\]

These are familiar from the action of symmetry transformations on a complex wavefunction or field operator in quantum mechanics and quantum field theory. If
\[
\psi \rightarrow \hat{U} \psi, \quad \psi^\dagger \rightarrow \psi^\dagger \hat{U}^+, \quad \hat{U}^+ \hat{U} = 1, \tag{8.2.19}
\]
then we have the following transformation properties for the bilinears:
\[
\psi^\dagger \psi \rightarrow \psi^\dagger \psi, \quad \text{invariant scalar},
\]
\[
\psi^\dagger M \psi \rightarrow \psi^\dagger (\hat{U}^+ \hat{M} \hat{U}) \psi, \quad \text{adjoint representation for } \text{Tr}(\hat{M}) = 0.
\]

If \( \psi_i \) \([\psi^\dagger_i] \) transforms in the \( (n + 1) \) \([\bar{n} + 1] \) representation of SU\((n + 1)\), then \( M^j_i \) can be decomposed into a linear combination of su\((n + 1)\) generators. The similarity transformation \( \hat{M} \rightarrow \hat{U} \hat{M} \hat{U} \) acts in the adjoint representation. In Sec. 8.5, we will connect tensors with different combinations of (anti)symmetrized upper and lower indices with more complex irreducible representations of su\((n + 1)\).

Eq. (8.2.19) gives an alternative way to define the \( \bar{n} + 1 \) representation. If we express the SU\((n + 1)\) transformation as
\[
\psi_i \rightarrow U_i^j \psi_j, \quad \hat{U} = \exp(i \hat{T}^a \theta^a), \tag{8.2.20}
\]
where \( \{ \hat{T}^a \} \) are the generators acting on the \( n + 1 \) representation, then

\[
(\psi^j)^t \to (\psi^j)^t(U^*)^j, \quad \hat{U}^* = \exp \left[ i(\hat{T}^a)^*(\theta^a)^* \right].
\]

(8.2.21)

We restrict the generators to the Cartan subalgebra, so that we can take \( \hat{T}^a \in \{ \hat{H}^{(n+1)}_{\hat{a}} \} \). Then \( \hat{T}^a = (\hat{T}^a)^\dagger \) and \( \theta^a \in \mathbb{R} \). Moreover, the \( \hat{H}^{(n+1)}_{\hat{a}} \) can be taken to be diagonal and hence symmetric (modulo 7). We conclude that the Cartan subalgebra generators of \( \hat{n + 1} \) are (c.f. Fig. 8.4)

\[
\hat{H}^{(n+1)}_{\hat{a}} = -\hat{H}^{(n+1)}_{\hat{a}}.
\]

(8.2.22)

Explicitly, if the weight vectors and weights of the defining \( n + 1 \) representation are

\[
|1\rangle = \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix} \quad \Leftrightarrow \quad \Lambda^{(1)} = (1, 0, 0, \cdots, 0, 0), \quad \text{highest weight state},
\]

\[
|2\rangle = \begin{bmatrix}
0 \\
1 \\
\vdots \\
0
\end{bmatrix} \quad \Leftrightarrow \quad \Lambda^{(2)} = (-1, 1, 0, \cdots, 0, 0),
\]

(8.2.23)

\[
|n + 1\rangle = \begin{bmatrix}
0 \\
0 \\
\vdots \\
1
\end{bmatrix} \quad \Leftrightarrow \quad \Lambda^{(n+1)} = (0, 0, 0, \cdots, 0, -1), \quad \text{lowest weight state},
\]

then the weight vectors and weights of the conjugate \( \bar{n + 1} \) representation are

\[
|1\rangle = \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix} \quad \Leftrightarrow \quad -\Lambda^{(1)} = (-1, 0, 0, \cdots, 0, 0), \quad \text{lowest weight state},
\]

\[
|2\rangle = \begin{bmatrix}
0 \\
1 \\
\vdots \\
0
\end{bmatrix} \quad \Leftrightarrow \quad -\Lambda^{(2)} = (1, -1, 0, \cdots, 0, 0),
\]

(8.2.24)

\[
|n + 1\rangle = \begin{bmatrix}
0 \\
0 \\
\vdots \\
1
\end{bmatrix} \quad \Leftrightarrow \quad -\Lambda^{(n+1)} = (0, 0, 0, \cdots, 0, 1), \quad \text{highest weight state}.
\]

Therefore the weights of \( \bar{n + 1} \) are minus those of \( n + 1 \); the highest-weight state of \( n + 1 \) corresponds to the lowest weight state of \( \bar{n + 1} \). This ensures that the weight of each term \( \{ U_1 V^1, U_2 V^2, \ldots, U_{n+1} V^{n+1} \} \) is zero, a necessary condition for \( U_i V^i \) to be a scalar.

Eq. (8.2.22) can be generalized to all irreducible representations. A representation \( \Lambda \) (defined by the Dynkin coefficients of its highest weight) is \textbf{self-conjugate} if its lowest weight is equal to minus its highest weight. Otherwise, the representation is said to be \textbf{complex}, and there exists a \textbf{conjugate} representation \( \Lambda^* \) which has weights equal to minus those of \( \Lambda \).
Figure 8.7: Weight trees for the three fundamental representations of $\text{su}(4)$: $(1,0,0) \Leftrightarrow \omega_1$ (defining “4”) $(0,1,0) \Leftrightarrow \omega_2$, and $(0,0,1) \Leftrightarrow \omega_1$ (conjugate “$\bar{4}$”). The simple root labels $\alpha_{1,2,3}$ indicate the difference between the weights above and below the connecting line. The $(0,1,0)$ representation is self-conjugate.

**Claim:** All representations of $B_n = \text{so}(2n + 1)$, $C_n = \text{sp}(2n)$, $G_2$, $F_4$, $E_7$, and $E_8$ are self-conjugate. This is related to the absence of reflection symmetry for the Dynkin diagrams of these algebras [2]. It turns out that all $D_{2n} = \text{so}(4n)$ representations are also self-conjugate. By contrast, $A_n = \text{su}(n + 1)$, $D_{2m+1} = \text{so}(4m + 2)$, and $E_6$ have both self-conjugate and pairwise conjugate representations.

### 8.2.4 The highest root for $A_n$, $B_n$, $C_n$, and $D_n$

With the basics of tensor-building and conjugate representations in place, we can now define the adjoint representation for each of the classical families in terms of the Dynkin coefficients for the highest root $\theta$. For $A_n = \text{su}(n + 1)$, the adjoint representation corresponds to a traceless product of $n + 1$ and $n + 1$ representations, $\tilde{T}_{ij}$ in Eq. (8.2.17b). Tracelessness does not force the highest-weight state $\tilde{T}^{n+1}_{n+1}$ to vanish [see Eqs. (8.2.23) and (8.2.24)]. Therefore

$$\theta = \omega_1 + \omega_n \Leftrightarrow (1,0,0,\ldots,0,1), \quad \text{Highest root of } A_n.$$  

(8.2.25)

For the orthogonal algebras $B_n = \text{so}(2n + 1)$ and $D_n = \text{so}(2n)$, the generators acting in the defining vector representation are completely antisymmetric matrices [Eq. (7.3.2)]. The completely antisymmetric tensor $T_{[ij]}$ has weight $\omega_2$:

$$\theta = \omega_2 \Leftrightarrow (0,1,0,\ldots,0,0), \quad \text{Highest root of } B_n \text{ and } D_n.$$  

(8.2.26)

Finally, in $C_n = \text{sp}(2n)$ the symmetric second rank tensor $T_{(ij)}$ is an irreducible representation, because it cannot be contracted with the $2n \times 2n$ antisymmetric “metric” $\epsilon^{ij}$ [Eq. (7.2.2)]. [Note that $\epsilon^{ij}$ is a block rank-2 Levi-Civita symbol; this should not be confused

Figure 8.8: Weight geometries for the first two fundamental representations of $\text{su}(4)$: $(1,0,0) \Leftrightarrow \omega_1$ (defining) and $(0,1,0) \Leftrightarrow \omega_2$. Weights correspond to the vertices of the polyhedra. The $(0,1,0)$ representation corresponds to the second rank antisymmetric tensor $T_{[ij]}$. The roots are indicated with arrows.
with the rank-2n tensor \( \epsilon^{i_1 i_2 \ldots i_2n} \). Both are group invariants for \( \text{Sp}(2n) \).] In the defining \( 2n \) representation, the symplectic generators in fact satisfy a generalized symmetry constraint, see Eq. (2A.3.5). We therefore claim that \( T_{(ij)} \) is in fact the adjoint representation,

\[
\theta = 2 \omega_1 \Leftrightarrow (2, 0, 0, \ldots, 0, 0), \quad \text{Highest root of } C_n. \tag{8.2.27}
\]

The Dynkin coefficients for the highest root of the exceptional algebras are given in Sec. 13.A of [2].

### 8.2.5 \( \text{su}(4) \)

The \( \text{su}(3) \) algebra exhibits most, but not all general characteristics of \( \text{su}(n+1) \) representations. The root system for \( \text{su}(4) \) was constructed in Sec. 7.1.1. Unlike \( \text{su}(3) \), the defining and conjugate representations do not exhaust the fundamental weights. In \( \text{su}(4) \), the second fundamental weight \( \omega_2 \Leftrightarrow (0, 1, 0) \) is associated to rank-2 antisymmetric tensor \( T_{ij} \). This representation is self-conjugate. The weight trees for all three fundamental representations are shown in Fig. 8.7. Using Eq. (8.1.8) to reconstruct the simple root expansion coefficients, one obtains the weight geometries shown in Fig. 8.8. The geometry of the defining \( \omega_1 \), conjugate \( \omega_3 \), and adjoint \( \theta = \omega_1 + \omega_3 \) representations were depicted in Fig. 7.3. The fundamental representations \( \{ \omega_m \} \) \( (1 \leq m \leq n) \) plus the adjoint representation \( \theta = \omega_1 + \omega_n \) exhaust the \( n+1 \) different conjugacy classes of \( \text{su}(n+1) \), as we discuss in Sec. 8.4, below.

- **Exercise:** Verify the weight geometries in Fig. (8.8).
- **Exercise:** Construct the weight tree for the \((0,2,0)\) representation of \( \text{su}(4) \).

### 8.3 \( \text{so}(2n+1) \), \( \text{so}(2n) \), and \( \text{sp}(2n) \)

We now explore fundamental highest-weight representations for the other classical families. At rank two, we have the special cases \( D_2 = A_1 \times A_1 \) and \( B_2 = C_2 \) (Table 1 in module 7). The representations of the former are trivial, while those of the latter can be inferred from \( C_n \) at higher ranks. We therefore begin with rank three.

#### 8.3.1 \( \text{so}(2n+1) \)

Consider \( \text{so}(7) = B_3 \). The defining vector representation is \((1,0,0) \Leftrightarrow \omega_1 \). The adjoint representation (second-rank antisymmetric tensor \( T_{ij} \)) is \((0,1,0) \Leftrightarrow \omega_2 \). Can we associate \((0,0,1) \Leftrightarrow \omega_3 \) with a third-rank antisymmetric tensor? To determine this, we compute the weight tree for \((1,0,0)\) using the Cartan matrix [Eq. (7.4.20)]

\[
\hat{A}_{B_3} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix} \tag{8.3.1}
\]

The results are shown in Fig. 8.9.

As in \( A_n \), all completely antisymmetric tensors are irreducible representations of \( B_n \) or \( D_n \), since we cannot trace out any pair of indices with the symmetric “metric” \( \delta^{ij} \) (i.e. the usual vector dot product). We will compute the Dynkin coefficients for the antisymmetric tensor representations of \( \text{so}(7) \) from the \((1,0,0)\) weight tree, as we did in Eq. (8.2.6).

- **m = 2:** second-rank, antisymmetric tensor \( T_{ij} \) with 21 components. From Fig. 8.9, this has highest weight

\[
(1,0,0) + (-1,1,0) = (0,1,0) \rightarrow \omega_2, \quad \tag{8.3.2a}
\]

consistent with the identification of \((0,1,0)\) with the adjoint representation.

- **m = 3:** \( T_{ijkl} \) with 35 components. The highest weight is

\[
(1,0,0) + (-1,1,0) + (0,-1,2) = (0,0,2) \rightarrow 2\omega_3. \quad \tag{8.3.2b}
\]

Thus the third-rank antisymmetric tensor is associated to \((0,0,2) \rightarrow 2\omega_3\), and is *not* a fundamental weight. Note that there is no need to consider higher-rank antisymmetric tensors, since these are equivalent to the first three via the rank-7 Levi-Civita symbol:

\[
\begin{align*}
T_{[i_1i_2i_3]} &\equiv \epsilon_{i_1i_2i_3i_4i_5i_6i_7} \hat{T}_{[i_1i_2i_3i_4i_5i_6]} \Leftrightarrow (0,0,2) \rightarrow 2\omega_3, \\
T_{[i_1i_2]} &\equiv \epsilon_{i_1i_2i_3i_4i_5i_6i_7} \hat{T}_{[i_1i_2i_3i_4i_5]} \Leftrightarrow (0,1,0) \rightarrow \omega_2, \\
T_{i_1} &\equiv \epsilon_{i_1i_2i_3i_4i_5i_6i_7} \hat{T}_{[i_1i_2i_3i_4i_5]} \Leftrightarrow (1,0,0) \rightarrow \omega_1, \\
T &\equiv \epsilon_{i_1i_2i_3i_4i_5i_6i_7} \hat{T}_{[i_1i_2i_3i_4i_5]} \Leftrightarrow 0 \text{ (scalar)}. \tag{8.3.3}
\end{align*}
\]
Figure 8.9: Weight trees for the first and last fundamental representations of so(7): (1,0,0) ⇔ ω₁ (defining) and (0,0,1) ⇔ ω₃. The simple root labels \( \overline{\alpha}_{1,2,3} \) indicate the difference between the weights above and below the connecting line. The (0,0,1) representation has \( 2^3 = 8 \) total states, and corresponds to the fundamental spinor representation of so(7). We will discuss spinor representations of \( B_n \) and \( D_n \) in detail in module 10. Like all representations of \( B_n \), these representations are self-conjugate.

In \( B_n \), \( C_n \), and \( D_n \), the defining representation is self-conjugate. There is thus no need to distinguish between upper and lower indices. (In fact all representations of \( B_n \), \( C_n \), and \( D_{2m} \) are self-conjugate, but this is not true for \( D_{2m+1} \).

The question remains: what is the (0,0,1) → ω₃ representation of so(7) = \( B_3 \)? Clearly it cannot be obtained by any combination of symmetrizing or antisymmetrizing vector indices. The weight tree is shown in Fig. 8.9. There are \( 2^3 = 8 \) different weights in (0,0,1); all turn out to be non-degenerate. The highest weight representation \( ω₃ \) is the fundamental spinor representation of so(7). We will discuss spinor representations of \( B_n \) and \( D_n \) in detail in module 10.

Using the Eq. (8.1.8) and the root geometry for \( B_3 \), one can determine the weight geometries for the fundamental vector (1,0,0) and spinor (0,0,1) representations. These are depicted in Fig. 8.10. An interesting and important fact is that the positive weights are orthonormal [c.f. Eqs. (8.3.4) and (8.3.5)]. Note that roots are not depicted in either subfigure.
weights of the vector representation (Fig. 8.9)

\[
\begin{align*}
\Lambda^{(1)} &= (1, 0, 0) = \omega_1, \\
\Lambda^{(2)} &= (-1, 1, 0), \\
\Lambda^{(3)} &= (0, -1, 2), \\
\end{align*}
\] (8.3.4)

are orthonormal:

\[
\langle \Lambda^{(a)}, \Lambda^{(b)} \rangle = \delta^{a,b},
\] (8.3.5)

(adopting a standard normalization of the roots such that \(|\alpha|^2 = 2\) for long roots—see module 9). In Eqs. (8.3.4) and (8.3.5), we define positive weights in the vector representation as the highest weight and its descendants above the central zero weight.

**Exercise:** Prove Eq. (8.3.5) using the Dynkin coefficients for each weight and the quadratic form matrix \(F_{ij}\) for \(so(7)\), given by Eq. (8.6.2b).

By contrast, the weights of the fundamental spinor representation (\(\equiv \{\Xi^{(a)}\}\)) correspond to the vertices of the cube,

\[
\Xi^{(a)} \in \frac{1}{2}(s_1\Lambda^{(1)} + s_2\Lambda^{(2)} + s_3\Lambda^{(3)}),
\] (8.3.6)

where \(s_{1,2,3} \in \pm 1\). See Fig. 8.10. These facts generalize to \(B_n = so(2n + 1)\) for \(n \geq 3\):

<table>
<thead>
<tr>
<th>Highest fundamental weight representations of (B_n = so(2n + 1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>• The (n) positive weights ({\Lambda^{(a)} &gt; 0}) of the defining vector representation (\omega_1) are orthonormal [Eq. (8.3.5)].</td>
</tr>
<tr>
<td>• The fundamental weight (\omega_n) representation ((1 \leq m \leq n - 1)) corresponds to a rank-(m) antisymmetric tensor (T_{[i_1\ldots i_m]}).</td>
</tr>
<tr>
<td>• The fundamental weight (\omega_n) is the highest weight state of the fundamental spinor representation.</td>
</tr>
<tr>
<td>This has (2^n) non-degenerate weights that correspond to the vertices of an (n)-dimensional hypercube.</td>
</tr>
<tr>
<td>• The spinor representation cannot be obtained by tensoring together vector indices.</td>
</tr>
</tbody>
</table>

We will prove these statements in modules 9 and 10. Finally, we note that symmetric traceless tensors of vector indices are also irreducible representations:

\[
\tilde{T}_{(i_1\ldots i_m)}, \quad \tilde{T}_{(i_1i_2\ldots i_m)} = 0, \quad m \geq 1: \text{ irreducible representation of } so(2n + 1) \text{ with highest weight } m\omega_1.
\] (8.3.8)

The weights that correspond to irreducible representations built entirely from vector indices reside in a single conjugacy class of \(B_n\). This includes any combination of the fundamental weights \(\{\omega_m\}\) with \(1 \leq m \leq n - 1\). The fundamental spinor representation resides in a separate conjugacy class. Higher-rank spinor- or mixed spinor- and vector-index tensors belong to one of these two classes.

### 8.3.2 \(sp(2n)\)

The defining \(2n\) representation of \(sp(2n)\) has highest weight \(\omega_1\). Using the Cartan matrix [Eq. (7.2.21)]

\[
\tilde{A}_{C_n} = \begin{bmatrix}
2 & -1 & 0 & \ldots & 0 & 0 \\
-1 & 2 & -1 & \ldots & 0 & 0 \\
0 & -1 & 2 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & \ldots & -2 & 0 \\
\end{bmatrix},
\] (8.3.9)

we can obtain the weight tree shown in Fig. 8.11.

**Exercise:** Derive the weight tree shown in Fig. 8.11.

Unlike \(A_n\), \(B_n\), and \(D_n\), an antisymmetrized tensor \(T_{[i_1i_2\ldots i_m]}\) does not automatically correspond to an irreducible representation of \(C_n\). This is because any pair of indices can be contracted with the rank-2 antisymmetric “metric" \(\epsilon_{ij}\) [Eq. (7.2.2)], which is a group invariant. Here we disregard the distinction between upper and lower indices, because the defining and all other irreducible representations of \(sp(2n)\) are self-conjugate. We can define traceless antisymmetric tensors

\[
\tilde{T}_{[i_1\ldots i_m]}, \quad \epsilon_{pq} \tilde{T}_{[pqi_1\ldots i_m]} = 0, \quad 1 \leq m \leq n,
\] (8.3.10)
and these do correspond to irreducible representations. There is no need to consider a rank \( m > n \) antisymmetric tensor, since if it is non-zero it is equivalent to a rank \( 2n - m < n \) tensor, obtained by contracting indices with the rank-2\( n \) Levi-Civita symbol. The rank-\( m \) traceless tensor \( \tilde{T}_{[i_1 \cdots i_m]} \) has

\[
\frac{(2n)!}{m!(2n-m)!} - \frac{(2n)!}{(m-2)!(2n-m+2)!} = 2(n+1-m)\frac{(2n+1)!}{m!(2n-m+2)!}
\]

independent components. On the left-hand-side of this equation, the second term is the number of constraints imposed by the trace condition \( \epsilon_{pq} \tilde{T}_{[pqi_1 \cdots i_m]} = 0 \).

We determine the Dynkin coefficients for traceless antisymmetric tensors as in Eq. (8.2.6).

- \( m = 2 \): \( \tilde{T}_{[ij]} \). Tracelessness does not require that \( \tilde{T}_{12} = -\tilde{T}_{21} \) is equal to zero. Fig. 8.11 therefore implies that the highest weight is

\[
(1, 0, 0, \ldots, 0) + (-1, 1, 0, \ldots, 0) = (0, 1, 0, \ldots, 0) \rightarrow \omega_2.
\]

(8.3.12a)

- \( m = 3 \): \( \tilde{T}_{[ijk]} \). The highest weight is

\[
(1, 0, 0, \ldots, 0) + (-1, 1, 0, \ldots, 0) + (0, -1, 1, 0, \ldots, 0) = (0, 0, 1, \ldots, 0) \rightarrow \omega_3.
\]

(8.3.12b)

- \( m = n \): \( \tilde{T}_{[i_1 i_2 \cdots i_n]} \). The highest weight is

\[
(1, 0, 0, \ldots, 0, 0, 0) + (-1, 1, 0, \ldots, 0, 0, 0) + \ldots + (0, 0, 0, \ldots, 0, -1, 1) = (0, 0, 0, \ldots, 0, 0, 1) \rightarrow \omega_n.
\]

(8.3.12c)

- \( m = n + 1 \): \( \tilde{T}_{[i_1 i_2 \cdots i_n i_{n+1}]} \). The weight is

\[
(0, 0, 0, \ldots, 0, 0, 1) + (0, 0, 0, \ldots, 0, 1, -1) = (0, 0, 0, \ldots, 0, 0, 1) \rightarrow \omega_{n-1}.
\]

(8.3.12d)

This is just the \( m = n - 1 \) tensor in disguise. More generally, the rank-\( 2n \) Levi-Civita symbol converts

\[
\epsilon_{[i_1 i_2 \cdots i_{n+m} j_1 \cdots j_{n-m}]} \tilde{T}_{[i_1 \cdots i_{n+m}]} \equiv \tilde{T}_{[j_1 \cdots j_{n-m}]}, \quad 1 \leq m \leq n.
\]

(8.3.12e)
As argued above in Sec. 8.2.4, completely symmetric tensors are automatically irreducible representations. We conclude that

\begin{align}
\text{Elementary tensor representations of } C_n = \text{sp}(2n): \notag \\
\bullet \quad \tilde{T}_{[i_1 i_2 \ldots i_m]}, \quad \epsilon_{p_q p_{p_q i_1 i_2 \ldots i_m}} = 0, \quad 1 \leq m \leq n : \text{irreducible representation with highest weight } \omega_m. \tag{8.3.13} \\
\bullet \quad T_{(i_1 i_2 \ldots i_m)}, \quad m \geq 1 : \text{irreducible representation with highest weight } m \omega_1. \notag
\end{align}

Thus, similar to \( \text{su}(n+1) \), all fundamental representations of \( C_n \) can be identified with (traceless) antisymmetric tensors. All irreducible representations of \( C_n \) can be formed by tensoring together copies of the defining \( \omega_1 \) representation, symmetrizing and/or antisymmetrizing, and removing all traces.

**Exercise:** Show that the top \( n \) weights \( \{ \Lambda^{(a)} \} \) of the defining \( 2n \) representation satisfy

\[
\langle \Lambda^{(a)}, \Lambda^{(b)} \rangle = \frac{1}{2} \delta^{a,b}. \tag{8.3.14}
\]

To do this, use the Dynkin coefficients for the weights shown in Fig. 8.11 and the explicit form of the quadratic form matrix \( F_{ij} \) given by Eq. (8.6.2c).

### 8.3.3 \( \text{so}(2n) \)

Finally, we consider \( D_n = \text{so}(2n) \). The defining \( 2n \)-component vector representation has highest weight \( \omega_1 \). The descendant weights obtain via the MWDF, using the Cartan matrix [Eq. (7.3.24)]

\[
\hat{A}_{D_n} = \begin{bmatrix}
2 & -1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & \ldots & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & \ldots & -1 & 2 & -1 & -1 \\
0 & 0 & 0 & \ldots & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & \ldots & 0 & -1 & 0 & 2
\end{bmatrix}. \tag{8.3.15}
\]

The result is the weight tree shown in Fig. 8.12.

As for \( B_n \), we expect there will be spinor representations of \( D_n \) corresponding to fundamental weights. To find them, we first search for completely antisymmetric tensor representations, following the same logic in Eqs. (8.2.6), (8.3.2), and (8.3.12). One finds that the first \( n-2 \) ranks correspond to fundamental weights; a traceless symmetric tensor of rank \( m \) corresponds as usual to weight \( m \omega_1 \):

\[
\text{Elementary tensor representations of } D_n = \text{so}(2n): \notag \\
\bullet \quad T_{[i_1 i_2 \ldots i_m]}, \quad 1 \leq m \leq n-2 : \text{irreducible representation with highest weight } \omega_m. \tag{8.3.16} \\
\bullet \quad \tilde{T}_{(i_1 i_2 \ldots i_m)}, \quad \delta_{p_q} \tilde{T}_{(p_q i_1 i_2 \ldots i_m)} = 0, \quad m \geq 1 : \text{irreducible representation with highest weight } m \omega_1. \notag
\]

As for \( B_n \) and \( C_n \), we do not distinguish upper and lower indices because the defining representation is self-conjugate.

What about rank-\((n-1)\) and rank-\(n\) tensors? The Dynkin labels for the highest weight state of \( T_{[i_1 \ldots i_{n-1}]} \) are given by [Eq. (8.3.16) and Fig. 8.12]

\[
\text{Elementary tensor representations of } D_n = \text{so}(2n): \notag \\
\bullet \quad T_{[i_1 i_2 \ldots i_{n-1}]} : \quad (0, 0, 0, \ldots, 0, 1, 0, 0) + (0, 0, 0, \ldots, 0, -1, 1, 1) = (0, 0, 0, \ldots, 0, 1, 1) \rightarrow \omega_{n-1} + \omega_n. \tag{8.3.17}
\]

For the rank-\(n\) tensor \( T_{[i_1 \ldots i_n]} \), there are two possibilities according to Fig. 8.12:

\[
\text{Elementary tensor representations of } D_n = \text{so}(2n): \notag \\
\bullet \quad T_{[i_1 i_2 \ldots i_n]} : \quad (0, 0, 0, \ldots, 0, 1, 1) + (0, 0, 0, \ldots, 0, -1, 1, 1) = (0, 0, 0, \ldots, 0, 0, 2) \rightarrow 2 \omega_n, \tag{8.3.18} \\
(0, 0, 0, \ldots, 0, 1, 1) + (0, 0, 0, \ldots, 0, -1, 1, 1) = (0, 0, 0, \ldots, 0, 2, 0) \rightarrow 2 \omega_{n-1}.
\]

We see that \( T_{[i_1 \ldots i_n]} \) splits into two representations. The interpretation is that there are really two independent rank-\(n\) antisymmetric tensors. Define

\[
\begin{align}
T_{[i_1 i_2 \ldots i_n]}^{(l)} &\equiv T_{[i_1 i_2 \ldots i_n]}, \\
T_{[i_1 i_2 \ldots i_n]}^{(f)} &\equiv \frac{1}{n!} \epsilon_{i_1 i_2 \ldots i_n j_1 \ldots j_n} T_{[j_1 j_2 \ldots j_n]}. \tag{8.3.19}
\end{align}
\]

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Note that

$$\frac{1}{n!} \epsilon_{i_1 \cdots i_n j_1 \cdots j_n} T^{(I)}_{[j_1 j_2 \cdots j_n]} = \left( \frac{1}{n!} \right)^2 \epsilon_{i_1 \cdots i_n j_1 \cdots j_n} \epsilon_{j_1 \cdots j_n k_1 \cdots k_n} T^{(I)}_{[k_1 k_2 \cdots k_n]} = T^{(I)}_{[i_1 i_2 \cdots i_n]},$$

(8.3.20)

$$= T^{(I)}_{[i_1 i_2 \cdots i_n]},$$

(8.3.21)

where we have used the fact (not proven here) that

$$\epsilon_{i_1 \cdots i_n j_1 \cdots j_n} \epsilon_{j_1 \cdots j_n k_1 \cdots k_n} = n! \sum_P (-1)^P \delta_{i_1, P(k_1)} \delta_{i_2, P(k_2)} \cdots \delta_{i_n, P(k_n)},$$

and $P$ denotes a permutation of the $n$ symbols $\{k_1, k_2, \ldots, k_n\}$. Thus $T^{(I)}$ and $T^{(II)}$ are “dual” to each other. One can form the combinations [4]

$$T^{\pm}_{[i_1 i_2 \cdots i_n]} = T^{(I)}_{[i_1 i_2 \cdots i_n]} \pm T^{(II)}_{[i_1 i_2 \cdots i_n]},$$

(8.3.22)

Then $T^+$ ($T^-$) is self-dual (anti-self-dual), and these are independent representations.

Therefore the rank $n - 1$ and $n$ antisymmetric tensors do not correspond to fundamental weights. In $D_n$, there are two fundamental spinor representations. We will prove the following statements in module 10:

<table>
<thead>
<tr>
<th>Fundamental spinor representations of $D_n = \mathfrak{so}(2n)$:</th>
</tr>
</thead>
<tbody>
<tr>
<td>• The fundamental weights $\omega_{n-1}$ and $\omega_n$ (associated to the “ears” of the $D_n$ Dynkin diagram in Fig. 8.3) are the highest weight states of the two fundamental spinor representations.</td>
</tr>
<tr>
<td>• Each spinor representation has $2^{n-1}$ non-degenerate weights. These can be viewed as the bipartitioning of the $2^n$-dimension $\mathfrak{so}(2n + 1)$ fundamental spinor representation.</td>
</tr>
<tr>
<td>• The spinor representations cannot be obtained by tensoring together vector indices.</td>
</tr>
</tbody>
</table>

Although the fundamental spinor representations cannot be constructed by tensoring together vector indices, the converse is not true: all representations can be constructed by tensoring together fundamental spinor indices. The key to the construction is the Clifford algebra, introduced in module 10. The semi-simple algebra $D_2 = A_1 \times A_1 = \mathfrak{su}(2) \times \mathfrak{su}(2)$ gives an extreme example

```
 1 0 0 0 0 0 0 0 0
-1 1 0 0 0 0 0 0 0
 0 -1 0 0 0 0 0 0 0
 0 0 0 -1 1 0 0 0 0
 0 0 0 0 0 -1 1 0 0
 0 0 0 0 0 0 -1 1 0
 0 0 0 0 0 0 0 -1 -1
```

Figure 8.12: Weight tree for the defining representation of $\mathfrak{so}(2n)$ ($D_n$): $(1, 0, 0, \cdots, 0) \Leftrightarrow \omega_1$. The simple root labels $\{\alpha_i\}$ indicate the difference between the weights above and below the connecting line.
of this, since the two fundamental weights $\omega_{1,2}$ are spin-1/2 spinor representations. In high-energy (and recently, condensed matter) physics, these two inequivalent representations of $so(4)$ correspond to left- and right-handed Weyl fermions.

For $D_{2m} = so(4m)$ ($m \in \{1, 2, \ldots \}$), the two fundamental spinor representations are each self-conjugate. For $D_{2m+1} = so(4m + 2)$ ($m \in \{1, 2, \ldots \}$), the two fundamental spinor representations are pairwise conjugates. The spinor representations belong to different conjugacy classes than tensors built from vector indices, but the overall class structure of $D_n$ is slightly more involved than it is for the other algebras. It will be discussed in the next section.

**Exercise:** Show that the two spinor representations are self-conjugate (pairwise conjugate) for $so(8)$ [$so(10)$] by computing the weight trees from the Dynkin coefficients

a. $so(8)$: $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$,

b. $so(10)$: $(0, 0, 0, 1, 0)$ and $(0, 0, 0, 0, 1)$.

How many distinct weights do you find in each case?

### 8.4 Lattices and Conjugacy Classes

In a rank-$n$ Lie algebra, all roots can be expressed as linear combinations of $n$ simple roots with integral coefficients. The MWDF Eq. (5.1.23) then implies that all weights in a given irreducible representation consist of a section of an infinite $n$-dimensional lattice generated by the simple root vectors. The weight geometries computed thus far are displayed in Figs. 8.2, 8.5, and 8.6 [$A_2 = su(3)$], Figs. 7.3 and 8.8 [$A_3 = su(4)$], and Fig. 8.10 [$B_3 = so(7)$]. These figures make it clear that different representations of the same algebra can reside on different lattices. These differ only by a fractional lattice vector displacement relative to the origin, which we assign to the zero weight (i.e., the Cartan subalgebra of the adjoint representation).

In other words, the set of all representations divides into different conjugacy classes (or congruence classes), and these classes label the different lattices. We can crystallize this idea by defining two lattices:

- **The weight lattice:** $P = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \ldots + \mathbb{Z}\omega_n$.  
- **The root lattice:** $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \ldots + \mathbb{Z}\alpha_n$.  

All weights of all highest weight representations reside in the weight lattice $P$. Only those representations that include the zero weight reside in the root lattice $Q$. We label the conjugacy class of $Q$ by zero; this automatically includes the adjoint representation of any semi-simple Lie algebra. The lattices $P$ and $Q$ can be viewed as abelian groups, and the number of conjugacy classes is equal to the number of distinct elements of the coset $P/Q$, which is a finite (factor) group. The order $|P/Q|$ is equal to the determinant of the Cartan matrix. For the classical families, one has

| algebra | $|P/Q|$ |
|---|---|
| $A_n$ | $n + 1$ |
| $B_n$ | 2 |
| $C_n$ | 2 |
| $D_n$ | 4 |

(8.4.2)

We can assign conjugacy classes using the **congruence vector** $v$, an $n$-component vector with integer elements that is fixed for a given algebra. For a weight $\Lambda$, the conjugacy classes are defined via

$$\Lambda \cdot v \equiv \sum_{i=1}^{n} \Lambda_i v_i \mod |P/Q|.$$  

(8.4.3)

In this equation, the $\{\Lambda_i\}$ are the Dynkin coefficients of weight $\Lambda$.

#### 8.4.1 $A_n = su(n + 1)$

The congruence vector is

$$v = \{1, 2, 3, \ldots, n\},$$  

(8.4.4)

so that

$$\Lambda \cdot v = \Lambda_1 + 2\Lambda_2 + 3\Lambda_3 + \ldots + n\Lambda_n \mod (n + 1).$$  

(8.4.5)
Figure 8.13: su(3) representations in each of the algebra’s three conjugacy classes. Classes 0, 1, and 2 include the adjoint, defining 3, and conjugate $\bar{3}$ representations, respectively (see Fig. 8.2). Here we display the weights of three higher-dimensional representations: $(3,0)$, $(2,1)$, and $(2,0)$. All reside on triangular lattices generated by the simple root vectors $\alpha_1$ and $\alpha_2$, but the origin for the class 1 or 2 lattice is shifted relative to class 0. The adjoint origin (the zero weight associated to the Cartan subalgebra) is indicated by the axes drawn in each subfigure.

Thus the fundamental weight representation with highest weight $\omega_m$ ($1 \leq m \leq n$) resides in class $m$, while the adjoint representation resides in class 0. The defining $n + 1$ (conjugate $n - 1$) representation is in class 1 ($n$).

The three classes of su(3) were previously exhibited in Fig. 8.2. Fig. 8.13 shows three other representatives of the su(3) conjugacy classes.

**8.4.2 $B_n = \text{so}(2n + 1)$**

We first note that all irreducible representations of $B_n$ are self-conjugate, which means the lowest weight is equal to minus the highest weight in every representation. This in part explains the simpler conjugacy class structure relative to $A_n$.

The congruence vector is

$$v = \{0, 0, 0, \ldots, 0, 1\},$$

so that

$$\Lambda \cdot v = \Lambda_n \mod (2).$$

Recall that $\omega_n$ corresponds to the fundamental spinor representation. This resides in class 1. All other fundamental weight representations $\omega_m$ ($1 \leq m \leq n - 1$) reside in class 0. Moreover, all representations built from the first $n - 1$ fundamental weights also reside in class 0, as does any representation in which $\Lambda_n$ is even. These are the irreducible representations that obtain by tensoring together vector indices. Irreducible representations built from an even (odd) number of spinor indices and any number of vector indices reside in class 0 (1).

**8.4.3 $C_n = \text{sp}(2n)$**

All irreducible representations of $C_n$ are self-conjugate.

The congruence vector is

$$v = \{1, 2, 3, \ldots, n\},$$

so that

$$\Lambda \cdot v = \Lambda_1 + 2\Lambda_2 + 3\Lambda_3 + \ldots + n\Lambda_n \mod (2).$$

The odd (even) fundamental weight representation $\omega_m$ with $m \in \{1, 3, 5, \ldots\}$ ($m \in \{2, 4, 6, \ldots\}$) resides in class 1 (0). The adjoint representation $\theta = 2\omega_1$ resides in class 0 (as expected).
8.4.4 \( D_{2m} = \mathfrak{so}(4m), \ m \in \mathbb{Z}_+. \)

All irreducible representations of \( D_n \) with \( n = 2m \) (even) are self-conjugate. This leads to a simpler conjugacy class structure than \( D_{2m+1} \).

The congruence vector is

\[
v = \{0, 0, \ldots, 0, 1, 1\}, \tag{8.4.10}
\]

so that

\[
\Lambda \cdot v = \Lambda_{n-1} + \Lambda_n \mod (4), \quad n = 2m. \tag{8.4.11}
\]

The fundamental representations \( \{\omega_m\} \) with \( \{1 \leq m \leq n - 2\} \) reside in class 0. This includes the adjoint representation \( \omega_2 \). The fundamental spinor representations \( \omega_{n-1} \) and \( \omega_n \) reside in class 1. Completely antisymmetric tensors of rank \( n - 1 \) and \( n \) reside in class 2 [Eqs. (8.3.17) and (8.3.18)].

8.4.5 \( D_{2m+1} = \mathfrak{so}(4m + 2), \ m \in \mathbb{Z}_+. \)

Representations of \( D_n \) with \( n = 2m + 1 \) (odd) can be self- or pairwise-conjugate.

The congruence vector is

\[
v = \{2, 4, 6, \ldots, 2n - 4, n - 2, n\}, \quad n = 2m + 1. \tag{8.4.12}
\]

so that

\[
\Lambda \cdot v = 2\Lambda_1 + 4\Lambda_2 + 6\Lambda_3 + \ldots + 2(n - 2)\Lambda_{n-2} + (n - 2)\Lambda_{n-1} + n\Lambda_n \mod (4). \tag{8.4.13}
\]

Thus the fundamental highest weight representations divide up as follows:

<table>
<thead>
<tr>
<th>class ((2m - 1) \mod (4))</th>
<th>fundamental weight representations</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \omega_2, \omega_4, \omega_6, \ldots, \omega_{n-3} )</td>
</tr>
<tr>
<td>2</td>
<td>( \omega_1, \omega_3, \omega_5, \ldots, \omega_{n-2} )</td>
</tr>
<tr>
<td>(2m+1) \mod (4) ( \omega_{n-1} )</td>
<td></td>
</tr>
</tbody>
</table>

Completely antisymmetric tensors of even (odd) rank less than \( n - 2 \) reside in class 0 (2). Completely symmetric tensors of any even (odd) rank also reside in class 0 (2). The rank-(\(n - 1\)) completely antisymmetric tensor [Eq. (8.3.17)] resides in class 0, while the rank-\(n\) antisymmetric tensor [Eq. (8.3.17)] divides into two representations, both in class 2.

- **Exercise**: Prove two facts about the \((0,2,0)\) and \((0,0,2)\) partitions of the third-rank antisymmetric tensor of \( D_3 \):

  1. The representations \((0,2,0)\) and \((0,0,2)\) reside in the same conjugacy class (for example, share a weight),
  2. The \((0,2,0)\) and \((0,0,2)\) representations are pairwise conjugate.

To do this, compute the weight trees for these representations explicitly. Are there degenerate weights?

The fundamental spinor representations \( \{\omega_{n-1}, \omega_n\} \) respectively reside in conjugacy classes \( \{1, 3\} \) or \( \{3, 1\} \), depending on \( m \).

8.5 **Tensors and tableaux for \( \mathfrak{su}(n) \)**

For the four classical Lie algebra families, we have seen how to associate completely antisymmetric and completely symmetric tensors (with suitable traces removed in \( B_n, C_n, \) and \( D_n \)) with the fundamental weights \( \{\omega_m\} \), or simple combinations thereof. The orthogonal algebra \( B_n \) (\( D_n \)) has in addition one (two) fundamental spinor representation(s). We will consider spinor representations and their tensor products in module 10.

Completely antisymmetric, symmetric, or spinor tensors are the most frequently encountered representations in many-body quantum physics, be it high energy or condensed matter. These can arise e.g. as composite operators built from products of elementary boson or fermion fields with indices transforming in some physical symmetry space (spin, particle-hole, color, flavor, replica, etc.). They can also arise as order parameters for effective low-energy field theories with a spontaneously broken continuous symmetry. Nevertheless, occasionally one must confront more complicated representations corresponding to mixed symmetry tensors.
The standard method for dealing with generic tensor representations in $A_{n-1} = \text{su}(n)$ is Young tableaux. These also appear in the classification of representations of the symmetric permutation group $S_n$ (of order $n!$). An advantage of Young tableaux is that they can be used to find all states in a representation, including degeneracies; the MWDF algorithm that we have been using only determines the weights, but not the dimensionality of the associated weight spaces. In the next module, we will however develop a recursive algorithm (Freudenthal) for determining weight degeneracies for any representation of any semi-simple Lie algebra, as well as the total dimension of such a representation (Weyl).

A disadvantage of Young tableaux is that they are mainly useful for $\text{su}(n)$, whereas we have tried to emphasize tools that work the same way for all Lie algebras.

### 8.5.1 Young tableaux

A Young tableau$^4$ is a top- and left-justified array of boxes. The boxes are arranged into rows. If the $m$th row has $p_m$ boxes, then the $(m+1)$th row can have $p_m+1$ boxes with $0 \leq p_m+1 \leq p_m$. In other words, the number of boxes in each successive row must either decrease or stay the same.

Young tableaux enumerate the partitions of the positive integers (natural numbers) $\mathbb{N}$. A partition is a decomposition of a number $n \in \mathbb{N}$ into a sum over components $\{n_i \in \mathbb{N}\}$, such that $1 \leq n_i \leq n$. For example, there are 5 partitions of the number 4:

- $4 = 1 + 1 + 1 + 1$
- $= 1 + 1 + 2$
- $= 2 + 2$
- $= 1 + 3$
- $= 4$.

The set of all Young tableaux with $n$ boxes is in one-to-one correspondence with the partitions of $n$, see Fig. 8.14.

### 8.5.2 Young tableau for a generic highest weight representation; tensors

There is a canonical construction for assembling a Young tableau from a highest weight representation of $A_{n-1}$, using the expansion of the weight in terms of an orthonormal basis (instead of the simple roots, coroots, or fundamental weights). This can be found e.g. in Sec. 13.3.2 of [2]. The derivation leads to the conclusion that a highest weight can be viewed in terms of the integral partitions of its Dynkin labels. Because Young tableaux precisely enumerate partitions of the positive integers, they are an efficient tool for $\text{su}(n)$.

Here we will only state the result and provide examples of particular classes of tensor representations.

The Young tableau for a generic highest-weight representation of $\text{su}(n)$ is shown in Fig. 8.15. If the highest weight has Dynkin labels $\Lambda \rightarrow \{\Lambda_1, \Lambda_2, \ldots, \Lambda_{n-1}\}$, then the tableau has $\Lambda_{n-1} \geq 0$ columns of length $(n-1)$, $\Lambda_{n-2} \geq 0$ columns of length $(n-2)$, etc. We can interpret each box in the tableau as a particular index $i_m$ in a rank-$R$ tensor $(1 \leq m \leq R)$. All indices belong to the defining (“$n$”) representation $(1 \leq i_m \leq n)$. From Fig. 8.15, the rank $R$ is

$$R = \Lambda_1 + 2\Lambda_2 + 3\Lambda_3 + \ldots + (n-1)\Lambda_{n-1}$$

$$= \sum_{i=1}^{n-1} v_i \Lambda_i,$$

(8.5.1)

$^4$Unlabeled tableaux are sometimes called Young diagrams.
where \( v = \{1, 2, \ldots, n - 1\} \) is the congruence vector for \( \text{su}(n) \) [Eq. (8.4.4)]. Eq. (8.5.1) implies that all Young tableaux with the same number of boxes (mod \( n \)) belong to the same conjugacy class.

The rank-\( R \) tensor \( T_{i_1 i_2 \cdots i_R} \) corresponding to the tableau in Fig. 8.15 is made an irreducible representation by the following prescription:

1. Take a generic rank-\( R \) tensor, and symmetrize the indices within each row of the tableau.
2. Now antisymmetrize indices within each column.

A generic tensor of this form is obviously cumbersome to work with; in the remainder we will specialize to simpler cases.

The \( m \)th fundamental representation (rank-\( m \) fully antisymmetric tensor) with highest weight \( \Lambda = \omega_m \) corresponds to a single column of \( m \) boxes, (a) in Fig. 8.16. The rank-\( p \) fully symmetric tensor with highest weight \( \Lambda = p \omega_1 \) corresponds to a single row of \( p \) boxes. This displays the general scheme: indices in rows are (pre)symmetrized, while indices in columns are antisymmetrized. Moreover, we cannot have a column of length greater than \( n \). A length-\( n \) column corresponds to a set of \( n \) indices that can be contracted with the rank-\( n \) Levi-Civita symbol; for this reason, we always omit columns of length \( n \).

We can also exhibit tableaux for tensors built from conjugate \( \bar{n} \) ("upstairs") indices (see Sec. 8.2.3, above). The rank-\( m \) fully antisymmetric tensor \( T^{[i_1 \cdots i_m]} \) has weight \( \omega_{n-m} \), and thus corresponds to a single column of \( n - m \) boxes, (c) in Fig. 8.16. A fully symmetrized rank-\( p \) conjugate-index tensor \( T^{(i_1 \cdots i_p)} \) has highest weight \( \Lambda = (0, 0, \ldots, 0, p) = p \omega_m \). This corresponds to the \((n-1) \times p\) block tableau in Fig. 8.16(d). It is always important to remember that the boxes correspond to defining \( n \)-representation indices. In particular, the rank-1 conjugate representation \( T^i \) corresponds to a single maximal column of \((n-1)\) boxes.

More complicated tensors can be built by mixing defining-\( n \) and conjugate-\( \bar{n} \) indices. Two simple classes that correspond to irreducible representations are mixed, traceless, mutually (anti)symmetric tensors:

\[
\tilde{T}^{(j_2 \cdots j_q)}_{(i_1 \cdots i_p)}; \tilde{T}^{(i_1 \cdots i_p)}_{(j_2 \cdots j_q)} = 0: \text{bisymmetrized rank-} p \times \text{rank-} q \text{ traceless tensor. The highest weight is}
\]

\[
(p, 0, 0, \ldots, 0, q) \leftrightarrow p \omega_1 + q \omega_{n-1}.
\]

\[ (8.5.2) \]

Figure 8.16: Young tableau for simple tensors. (a) Fundamental representation \( T^{[i_1 \cdots i_m]} \) \( \leftrightarrow \omega_m \). (b) Fully symmetrized representation \( T_{(i_1 \cdots i_p)} \) \( \leftrightarrow p \omega_1 \). (c) Fully antisymmetrized rank-\( m \) tensor built from conjugate indices, \( T^{[i_1 \cdots i_m]} \) \( \leftrightarrow T^{[j_1 \cdots j_{n-m}]} \) \( \leftrightarrow \omega_{n-m} \). (d) Fully symmetrized rank-\( p \) tensor built from conjugate indices, \( T^{(i_1 \cdots i_p)} \) \( \leftrightarrow p \omega_{n-1} \).
Figure 8.17: Subfigures (a) and (b) can be associated to tensors with mixed lower and upper indices. Subfigure (a) corresponds to a symmetrized traceless tensor
\[ \tilde{T}_{(i_1 \cdots i_p)}^{(j_1 \cdots j_q)} \Leftrightarrow p \omega_1 + q \omega_{n-1} \] [Eq. (8.5.2)]. Subfigure (b) corresponds to an antisymmetrized traceless tensor
\[ \tilde{T}_{[i_1 \cdots i_p]}^{[j_1 \cdots j_q]} \Leftrightarrow \omega_p + \omega_{n-q} (p + q \neq n) \] [Eq. (8.5.3)]. Subfigure (c) shows a generic block tableau corresponding to
\[ p \omega_q \] [Eq. (8.5.4)].

- \( \tilde{T}_{[i_1 \cdots i_p]}^{[j_1 \cdots j_q]}; \tilde{T}_{[i_1 \cdots i_p]}^{[k_1 \cdots k_q]} = 0 \): biantisymmetrized rank-\( p \times \) rank-\( q \) traceless tensor. The highest weight is
\[ (0, 0, \ldots, 0, 1, 0, \ldots, 0, (n-q)^{th} \text{place}) \Leftrightarrow \omega_p + \omega_{n-q}, \quad (p + q \neq n). \] (8.5.3)

The corresponding tableaux correspond to the simple “gluing” of the defining rank-\( p \) and conjugate rank-\( q \) tableaux, Figs. 8.17(a,b). As a final example, we consider the tableau corresponding to the highest weight
\[ (0, 0, \ldots, 0, p^{th} \text{place}, 0, \ldots, 0, 0) \Leftrightarrow p \omega_q. \] (8.5.4)

The tableau is the “block” shown in Fig. 8.17(c). The general prescription says that we can view this irreducible representation as a tensor of \( p \times q \) defining indices, first symmetrized in the rows of the tableau, and finally antisymmetrized in the columns. I.e.,
\[ T_{[i_1 i_2 \cdots i_q,1][i_{1,2} i_{2,2} \cdots i_{q,2}] \cdots [i_{1,p} i_{2,p} \cdots i_{q,p}]} \] (8.5.5)

where \( i_{m,n} \) is an index appearing in the \( n^{th} \) row and \( m^{th} \) column of the tableau. Here the indices are grouped according to columns, each completely antisymmetric. The “presymmetrization” of indices sharing rows [i.e., \( (i_{1,1} i_{1,2} \cdots i_{1,p}) \)] is necessary to obtain an irreducible representation.

8.5.3 Dimension rule: “Factors over hooks”

We can encode any irreducible representation of \( \text{su}(n) \) by a Young tableau. Yet this doesn’t appear to give a particularly more transparent picture for a representation than its explicit tensor form (at least for simple cases). Nor do tableaux clearly convey geometrical information (e.g. the weight geometry of a representation). Why bother introducing them at all?

The main advantage is a number of simple tableau-based algorithms that allow efficient calculation of data pertaining to \( \text{su}(n) \) representations and their products. We will present a couple of these schemes here, and more later in the course. At this stage these are presented as “cookbook” recipes, rather than derived.

The dimension of an \( \text{su}(n) \) irreducible representation can be computed from its tableau using the “factors over hooks” rule. A hook is a line passing up through a particular column of a tableau up to some particular row, turning right, and passing out through that row. See Figs. 8.18(a)–(c).

To compute the dimension, there is the following prescription. Given a young tableau with \( R \) boxes,

1. Write “\( n \)” in the upper-left corner box.
2. Fill in rows with monotonically increasing sequences. I.e., the first row of length \( p \) should read \( \{n, n+1, n+2, \ldots, n+p\} \).
3. Fill in columns with monotonically decreasing sequences. I.e., the first column of length \( q \) should read \( \{n, n-1, n-2, \ldots, n-q+1\} \).
Figure 8.18: Ingredients in the “factors over hooks” rule for computing the dimension $d_{\Lambda}$ of an irreducible representation (labeled by the Dynkin coefficients $\{\Lambda_i\}$ if its highest weight state, or the associated tableau). (a)–(c) show examples of “hooks.” The hook in (a), (b), or (c) passes through 5, 4, or 3 boxes, respectively. Subfigures (d)–(f) [(g)] show $\text{su}(3)$ [su($n$)] tableaux enumerated according to the factors over hooks scheme. The dimensions are computed in the text.
4. The dimension $d_\Lambda$ of the representation is given by

$$d_\Lambda = \frac{\prod_{\text{boxes}} (\text{factor in each box})}{\prod_{\text{hooks}} (\text{number of boxes intersected by each hook})} = \frac{F_\Lambda}{H_\Lambda}. \quad (8.5.6)$$

This seemingly ad-hoc scheme is in fact a special case of the Weyl character formula, which we will derive for any Lie algebra in module 9.

Let us verify the factors over hooks rule for a number of cases. First we consider tableau for $\text{su}(3)$. Fig. 8.18(d) corresponds to the representation $\Lambda = 4\omega_1$, i.e. a rank-4 fully symmetric tensor. This has

$$d_{(4,0)} = \frac{(4 + 2)!}{4!2!} = 15$$

independent components. On the other hand, the numerator and denominator of Eq. (8.5.6) are given by

$$F_{(4,0)} = 3 \times 4 \times 5 \times 6, \quad H_{(4,0)} = 4 \times 3 \times 2 \times 1, \quad \Rightarrow d_{(4,0)} = 15, \quad (8.5.7)$$

in agreement with the above. Fig. 8.18(e) is the $\text{su}(3)$ adjoint representation with dimension $d_{(1,1)} = 8$. Eq. (8.5.6) gives

$$F_{(1,1)} = 3 \times 4 \times 2, \quad H_{(1,1)} = 3 \times 1 \times 1, \quad \Rightarrow d_{(1,1)} = 8. \quad (8.5.8)$$

Fig. 8.18(f) is the $\text{su}(3)$ representation $(2,1)$ with weight geometry shown in Fig. 8.6. We will show in module 9 that this has $d_{(2,1)} = 15$. Eq. (8.5.6) gives

$$F_{(2,1)} = 3 \times 4 \times 5 \times 2, \quad H_{(2,1)} = 4 \times 2 \times 1 \times 1, \quad \Rightarrow d_{(2,1)} = 15. \quad (8.5.9)$$

Finally, consider the $\text{su}(n)$ tableau shown in Fig. 8.18(g). This corresponds to the traceless biantisymmetric tensor $T_{[ij,\cdots]}^{qr} \iff \omega_p + \omega_{n-q}$, Eq. (8.5.3). The number of components of such a tensor is

$$d_{\omega_p + \omega_{n-q}} = \frac{n!}{p!(n-p)!} \frac{n!}{q!(n-q)!} \frac{n!}{(p+1)!(n-p+1)!} \frac{n!}{(q+1)!(n-q+1)!} \frac{n!}{(pq)!} \frac{n!(n+1)!}{p!(n-p+1)!} \frac{n!(n+1)!}{q!(n-q+1)!} \frac{n!(n+1)!(n-p-q+1)!}{p!(n-p+1)!q!(n-q+1)!}. \quad (8.5.10)$$

The second term on the right-hand-side of the first line of Eq. (8.5.10) accounts for the traceless constraint. On the other hand, Eq. (8.5.6) gives

$$F_{\omega_p + \omega_{n-q}} = \frac{n!(n+1)!}{q!(n-p+1)!}, \quad H_{\omega_p + \omega_{n-q}} = \frac{(n-q+1)!}{(n-p+1)!} (n-p-q)! \quad (8.5.11)$$

$\Rightarrow d_{\omega_p + \omega_{n-q}} = \frac{n!(n+1)!(n-p-q+1)!}{p!(n-p+1)!q!(n-q+1)!}.$

### 8.5.4 Semistandard Young tableaux: enumerating the states

We can also generate all states of an irreducible $\text{su}(n)$ representation explicitly using **semistandard** Young tableaux. From a given tableau corresponding to a particular highest weight representation, states are enumerated by filling in boxes with different sequences of positive integers. The prescription is to generate all possible labelings adhering to the following rules. Let $c_{i,j}$ be the integer appearing in the box in the $i^{th}$ row and $j^{th}$ column. Then

$$1 \leq c_{i,j} \leq n, \quad c_{i,j} \leq c_{i,j+1}, \quad c_{i,j} < c_{i+1,j}. \quad (8.5.12)$$
I.e., each successive element of a row is greater than or equal to the previous entry, while successive elements of a column are strictly greater than previous entries.

We apply this algorithm to the defining 3, conjugate \( \bar{3} \), and adjoint representations of \( \text{su}(3) \) in Fig. 8.19.

Each semistandard tableau corresponds to a particular state in the representation, irrespective of the dimension of the associated weight space. The weights themselves can be reconstructed as follows. For an \( \text{su}(n) \) semistandard tableau, a box with number \( p \) \((1 \leq p \leq n)\) is associated to the weight

\[
\omega_p - \omega_{p-1}, \quad \omega_0 = \omega_n = 0.
\]

The weight of the semistandard tableau obtains by summing the contributions of all boxes, c.f. Fig. 8.19 for \( \text{su}(3) \) examples.

### 8.6 Dual Coxeter numbers and quadratic form matrices

Here we list the quadratic form matrices and dual Coxeter numbers for the four classical Lie algebra families. In this section we use a standard normalization scheme for the root scalar products, such that long roots are normalized to 2: \( \langle \alpha, \alpha \rangle = 2 \). This means that roots and coroots are identical for simply laced algebras. This normalization scheme is different from the one employed to define the classical Lie algebras in module 7, and will be explained in module 9.

The dual Coxeter numbers are

\[
\begin{array}{ccc}
\text{algebra} & \text{group dimension} & g \\
A_n & n(n + 2) & n + 1 \\
B_n & n(2n + 1) & 2n - 1 \\
C_n & n(2n + 1) & n + 1 \\
D_n & n(2n - 1) & 2n - 2
\end{array}
\]

Figure 8.19: Semistandard tableaux for \( \text{su}(3) \) representations (a) \((1,0) (3)\), (b) \((0,1) (\bar{3})\), and (c) \((1,1) (\text{adjoint})\).
The quadratic form matrix [Eq. (8.1.5c)] for each of these families is given by

\[
\begin{align*}
\hat{F}_{A_n} &= \frac{1}{n+1} \begin{bmatrix}
    n & n-1 & n-2 & \cdots & 2 & 1 \\
n-1 & 2(n-1) & 2(n-2) & \cdots & 4 & 2 \\
n-2 & 2(n-2) & 3(n-2) & \cdots & 6 & 3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2 & 4 & 6 & \cdots & 2(n-1) & n-1 \\
1 & 2 & 3 & \cdots & n-1 & n \\
\end{bmatrix}, \\
\hat{F}_{B_n} &= \frac{1}{2} \begin{bmatrix}
    2 & 2 & \cdots & 2 & 1 \\
    2 & 4 & \cdots & 4 & 2 \\
    2 & 4 & 6 & \cdots & 6 & 3 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    2 & 4 & 6 & \cdots & 2(n-1) & n-1 \\
    1 & 2 & 3 & \cdots & n-1 & n/2 \\
\end{bmatrix}, \\
\hat{F}_{C_n} &= \frac{1}{2} \begin{bmatrix}
    1 & 1 & \cdots & 1 & 1 \\
    1 & 2 & 2 & \cdots & 2 & 2 \\
    1 & 2 & 3 & \cdots & 3 & 3 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    1 & 2 & 3 & \cdots & n-1 & n-1 \\
    1 & 2 & 3 & \cdots & n-1 & n \\
\end{bmatrix}, \\
\hat{F}_{D_n} &= \frac{1}{2} \begin{bmatrix}
    \min(i,j), & 1 \leq i, j \leq n-2, \\
    \min(i,j)/2, & 1 \leq i \leq n-2 \text{ and } j \in \{n-1, n\}, \text{ or } 1 \leq j \leq n-2 \text{ and } i \in \{n-1, n\}, \\
    (n-2)/4, & i = n-1 \text{ and } j = n, \text{ or } j = n-1 \text{ and } i = n, \\
    n/4, & i = j = n-1, \text{ or } i = j = n, \\
    2 & 2 & \cdots & 2 & 1 & 1 \\
    2 & 4 & 4 & \cdots & 4 & 2 & 2 \\
    2 & 4 & 6 & \cdots & 6 & 3 & 3 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    2 & 4 & 6 & \cdots & 2(n-2) & n-2 & n-2 \\
    1 & 2 & 3 & \cdots & n-2 & n/2 & (n-2)/2 \\
    1 & 2 & 3 & \cdots & n-2 & (n-2)/2 & n/2 \\
\end{bmatrix}.
\end{align*}
\]

References