9. Casimir operators, characters, dimension and strange formulae
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Contents

The discussion here follows chapters XI–XIII of [1] and Secs. 13.2 and 13.4 of [2].

9.1 Quadratic Casimir operator

An essential datum for each irreducible su(2) representation is the eigenvalue $C^{(2j)} = j(j+1)$ of the **quadratic Casimir operator** (see Sec. 1.3.3),

$$\hat{C} \equiv \hat{T}_z \hat{T}_z + \frac{1}{2}\left(\hat{T}_+ \hat{T}_- + \hat{T}_- \hat{T}_+\right), \tag{9.1.1}$$

where $2j \in \{0, 1, 2, \ldots\}$ is the Dynkin label for the spin-j representation (i.e., the Dynkin coefficient of the highest weight). The key properties of the Casimir operator are

1. It gives the same eigenvalue for all states in an irreducible representation,

$$\hat{C}\,|j,m\rangle = j(j+1)\,|j,m\rangle\,, \quad \hat{T}_z\,|j,m\rangle = m\,|j,m\rangle\,, \quad m \in \{-j, -j+1, \ldots, j-1, j\}. \tag{9.1.2a}$$

2. It commutes with all generators,

$$[\hat{C}, \hat{T}_+] = [\hat{C}, \hat{T}_-] = [\hat{C}, \hat{T}_z] = 0. \tag{9.1.2b}$$

Eigenvalues of an su(2)-invariant Hamiltonian can depend only on $C^{(2j)} = j(j+1)$, not on the "magnetic quantum number" m that distinguishes different weight vectors. Although each representation is uniquely labeled by its Casimir eigenvalue, it is important to remember that $\hat{C}$ is not defined abstractly for the Lie algebra $A_1 = \text{su}(2)$, but only for a particular representation in terms of the explicit bilinear sum of matrix generators in Eq. (9.1.1).[1]

For a generic rank-n Lie algebra, it turns out that one can construct exactly n Casimir operators satisfying the axioms in Eq. (9.1.2) [3]. We will discuss only the generalization of the quadratic Casimir in Eq. (9.1.1), which is unique. The other higher order Casimir operators are built from sums of m-fold products, where $m \geq 3$ is the order of each factor in the operator.[2]

We define the quadratic Casimir operator for a rank-n Lie algebra L as an operator acting on states of a generic finite-dimensional, irreducible representation as follows:

$$\hat{C} \equiv \sum_{i,j=1}^{n} \mathcal{A}_{ij}^{-1} \hat{H}_{\overline{\alpha}_i} \hat{H}_{\overline{\alpha}_j} + \sum_{\alpha \neq 0} \frac{\hat{E}_\alpha \hat{E}_{-\alpha}}{(e_\alpha, e_{-\alpha})}, \qquad \text{Quadratic Casimir operator,} \tag{9.1.3a}$$

$$\mathcal{A}_{ij} \equiv \langle \overline{\alpha}_i, \overline{\alpha}_j \rangle = A_{ij} \frac{|\overline{\alpha}_j|^2}{2}. \tag{9.1.3b}$$

Just as in su(2) [Eq. (9.1.1)], the operator $\hat{C}$ is a sum of bilinears of Cartan subalgebra elements $\{\hat{H}_{\overline{\alpha}_i}\}$ or root vector raising and lowering operators $\{\hat{E}_\alpha, \hat{E}_{-\alpha}\}$. The overlap matrix $\mathcal{A}_{ij}$ is a symmetrized version of the Cartan matrix $A_{ij} = \langle \overline{\alpha}_i, \overline{\alpha}_j^\vee \rangle$ [Eq. (5.3.8)]. Note that the sum over root vector operators in $\hat{C}$ includes all nonzero roots, not just positive or simple ones. The number of terms in the sum Eq. (9.1.3a) is equal to N_θ, the dimension of the adjoint representation: [recall that θ labels the highest weight (root) of the adjoint representation, Secs. 5.3.1 and 8.2.4]

$$N_\theta = n + |\Delta| = n + |\Delta_+| + |\Delta_-| = d. \tag{9.1.4}$$

In this equation, n denotes the rank, $|\Delta|$ ($|\Delta_\pm| = |\Delta|/2$) is the number of roots (number of positive or negative roots), and d is the dimension of the corresponding continuous group G generated by the Lie algebra L.
Claim I: The quadratic $\hat{C}$ is a pure number in any irreducible representation. I.e., it commutes with all $\{\hat{H}_{\overline{\alpha}_i}, \hat{E}_\alpha\}$.
Before we prove this statement, we examine some of its consequences.

1. Trace in the adjoint representation. Consider the trace

$$(C) \equiv \mathsf{Tr}\left[\hat{C}\right]_\theta, \tag{9.1.5}$$

where $\mathsf{Tr}[\cdots]_\Lambda$ denotes the trace in representation Λ; here θ indicates the adjoint representation. Since the trace in the adjoint representation is the Killing form $\mathsf{Tr}[\widehat{a}d_x\, \widehat{a}d_y]_\theta = (x, y)$, we have

$$(C) = \sum_{i,j=1}^{n} \mathcal{A}_{ij}^{-1} \left(h_{\overline{\alpha}_i}, h_{\overline{\alpha}_j}\right) + \sum_{\alpha \neq 0} = \sum_{i,j=1}^{n} \mathcal{A}_{ij}^{-1} \mathcal{A}_{ji} + \sum_{\alpha \neq 0} = N_\theta. \tag{9.1.6}$$

2. Action on the highest-weight state ϕ_Λ of an irreducible representation. Let Λ denote the associated highest weight. Then

$$\begin{aligned} \hat{C}\phi_\Lambda &= \sum_{i,j=1}^{n} \langle \Lambda, \overline{\alpha}_i \rangle \mathcal{A}_{ij}^{-1} \langle \overline{\alpha}_j, \Lambda \rangle \phi_\Lambda + \sum_{\alpha > 0} \left[\frac{\hat{E}_{-\alpha} \hat{E}_\alpha}{(e_\alpha, e_{-\alpha})} + \hat{H}_\alpha\right] \phi_\Lambda \\ &= \left[\langle \Lambda, \Lambda \rangle + \sum_{\alpha > 0} \langle \Lambda, \alpha \rangle\right] \phi_\Lambda. \end{aligned} \tag{9.1.7}$$

On the first line of Eq. (9.1.7), we have restricted the sum over root vectors to positive roots (because ϕ_Λ is a highest weight state annihilated by all $\hat{E}_{-\alpha}$ with $\alpha < 0$), and we have employed the definition for h_α in Eq. (4.4.10). On the second line, we have removed the resolution of the identity in the rank-n space of weights H^*. For a linearly independent, but not necessarily orthonormal basis of states $\{|i\rangle\}$, the identity can be expressed as

$$\hat{1}_n = \sum_{i,j=1}^{n} \mathcal{O}_{ij}^{-1} \left|i\right\rangle \left\langle j\right|, \qquad \mathcal{O}_{ij} = \left\langle i \middle| j \right\rangle. \tag{9.1.8}$$

[1] To write Eq. (9.1.1), we need to define the (matrix) product between generators. The only product operation defined for the Lie algebra itself is the Lie bracket, which becomes a matrix commutator in a finite-dimensional representation.

[2] The order $m+1$ of the $(m+1)$-fold products summed in a particular Casimir is called the *exponent* m. For each algebra A_n, B_n, C_n, and D_n, the n **exponents of the algebra** are tabulated (e.g.) in Sec. 13A of [2].

It will prove useful to rewrite Eq. (9.1.7) in a more compact way. We define the **Weyl vector** for a Lie algebra as the weight

$$\rho_{\mathsf{W}} \equiv \frac{1}{2}\sum_{\alpha>0}\alpha = \sum_{j=1}^{n}\omega_j, \qquad \text{Weyl vector for a rank-}n\text{ Lie algebra.} \tag{9.1.9}$$

The second equality implies that the Dynkin coefficients of the Weyl vector are $(\Lambda_1, \ldots, \Lambda_n) = (1, 1, 1, \ldots, 1)$. We will prove this in Sec. 9.3, below. Eq. (9.1.7) then becomes

$$\hat{C}\phi_\Lambda \equiv C^\Lambda\,\phi_\Lambda = \langle \Lambda, \Lambda + 2\rho_{\mathsf{W}}\rangle\,\phi_\Lambda. \tag{9.1.10}$$

<u>Proof:</u> The quadratic $\hat{C}$ commutes with all generators $\{\hat{H}_{\overline{\alpha}_i}, \hat{E}_\alpha\}$ in an irreducible representation. Exploiting the Lie bracket structure of a generic Lie algebra [Eq. (4.4.11)], we have

$$\begin{aligned}
\left[\hat{C}, \hat{H}_{\overline{\alpha}_i}\right] &= \sum_{\alpha\neq 0}\frac{\hat{E}_\alpha\left[\hat{E}_{-\alpha}, \hat{H}_{\overline{\alpha}_i}\right] + \left[\hat{E}_\alpha, \hat{H}_{\overline{\alpha}_i}\right]\hat{E}_{-\alpha}}{(e_\alpha, e_{-\alpha})} \\
&= \sum_{\alpha\neq 0}\frac{\hat{E}_\alpha\,\hat{E}_{-\alpha}\langle\alpha,\overline{\alpha}_i\rangle - \langle\alpha,\overline{\alpha}_i\rangle\hat{E}_\alpha\,\hat{E}_{-\alpha}}{(e_\alpha, e_{-\alpha})} = 0,
\end{aligned} \tag{9.1.11a}$$

$$\begin{aligned}
\left[\hat{C}, \hat{E}_\beta\right] &= \sum_{i,j=1}^{n}\mathcal{A}_{ij}^{-1}\langle\beta,\overline{\alpha}_j\rangle\left(\hat{H}_{\overline{\alpha}_i}\,\hat{E}_\beta + \hat{E}_\beta\,\hat{H}_{\overline{\alpha}_i}\right) + \sum_{\alpha\neq 0}\frac{\hat{E}_\alpha\left[\hat{E}_{-\alpha}, \hat{E}_\beta\right] + \left[\hat{E}_\alpha, \hat{E}_\beta\right]\hat{E}_{-\alpha}}{(e_\alpha, e_{-\alpha})} \\
&= \sum_{i,j=1}^{n}\mathcal{A}_{ij}^{-1}\langle\beta,\overline{\alpha}_j\rangle\left(\hat{H}_{\overline{\alpha}_i}\,\hat{E}_\beta + \hat{E}_\beta\,\hat{H}_{\overline{\alpha}_i}\right) + \sum_{\substack{\alpha\neq 0\\ \alpha\neq\beta}}\frac{N_{-\alpha,\beta}\,\hat{E}_\alpha\,\hat{E}_{\beta-\alpha}}{(e_\alpha, e_{-\alpha})} + {\color{red}\sum_{\substack{\alpha\neq 0\\ \alpha\neq -\beta}}\frac{N_{\alpha,\beta}\,\hat{E}_{\alpha+\beta}\,\hat{E}_{-\alpha}}{(e_\alpha, e_{-\alpha})}} - \hat{E}_\beta\,\hat{H}_\beta - \hat{H}_\beta\,\hat{E}_\beta.
\end{aligned} \tag{9.1.11b}$$

Next we invoke invariance of the Killing form [Eq. (4.4.8)], as well as Eq. (4.4.5) to obtain

$$\begin{aligned}
(e_\alpha, [e_\beta, e_\gamma]) &= N_{\beta,\gamma}\,(e_\alpha, e_{-\alpha})\,\delta_{\beta+\gamma+\alpha,0} = -N_{-\beta-\alpha,\beta}\,(e_\alpha, e_{-\alpha})\,\delta_{\beta+\gamma+\alpha,0} \\
&= ([e_\alpha, e_\beta], e_\gamma) = N_{\alpha,\beta}\,(e_{\alpha+\beta}, e_{-\alpha-\beta})\,\delta_{\alpha+\beta+\gamma,0}, \\
\Rightarrow \frac{N_{\alpha-\beta,\beta}}{(e_{\beta-\alpha}, e_{\alpha-\beta})} &= -\frac{N_{-\alpha,\beta}}{(e_\alpha, e_{-\alpha})}.
\end{aligned} \tag{9.1.12}$$

In this equation, we have used $N_{\alpha,\beta} = -N_{\beta,\alpha}$, which follows from the antisymmetric property of the Lie bracket [Eq. (1.2.1a)]; on the last line we have sent $\alpha \to -\alpha$ on both sides. We can therefore rewrite the red term in Eq. (9.1.11b) as follows:

$$\sum_{\substack{\alpha\neq 0\\ \alpha\neq -\beta}}\frac{N_{\alpha,\beta}\,\hat{E}_{\alpha+\beta}\,\hat{E}_{-\alpha}}{(e_\alpha, e_{-\alpha})} = \sum_{\substack{\alpha'\neq 0\\ \alpha'\neq\beta}}\frac{N_{\alpha'-\beta,\beta}\,\hat{E}_{\alpha'}\,\hat{E}_{\beta-\alpha'}}{(e_{\alpha'-\beta}, e_{-\alpha'+\beta})} = -\sum_{\substack{\alpha'\neq 0\\ \alpha'\neq\beta}}\frac{N_{-\alpha',\beta}\,\hat{E}_{\alpha'}\,\hat{E}_{\beta-\alpha'}}{(e_{\alpha'}, e_{-\alpha'})}, \tag{9.1.13}$$

where we have defined $\alpha' \equiv \alpha + \beta$. The two terms in Eq. (9.1.11b) explicitly involving the structure constants therefore exactly cancel, leaving

$$\left[\hat{C}, \hat{E}_\beta\right] = \hat{E}_\beta\left[\sum_{i,j=1}^{n}\mathcal{A}_{ij}^{-1}\langle\beta,\overline{\alpha}_j\rangle\hat{H}_{\overline{\alpha}_i} - \hat{H}_\beta\right] + \left[\sum_{i,j=1}^{n}\mathcal{A}_{ij}^{-1}\langle\beta,\overline{\alpha}_j\rangle\hat{H}_{\overline{\alpha}_i} - \hat{H}_\beta\right]\hat{E}_\beta = 0. \tag{9.1.14}$$

The final equality follows from the representation of $\hat{H}_\beta$ in the linearly independent basis $\{H_{\overline{\alpha}_j}\}$; e.g., consider a state $|f\rangle$ multiplied by the resolution of the identity in Eq. (9.1.8):

$$|f\rangle = \sum_{i,j=1}^{n}\mathcal{O}_{ij}^{-1}\,|i\rangle\,\langle j\,|f\rangle\,.$$

Q.E.D.

9.1.1 Standard normalization for Killing form and Casimir

The Dynkin coefficients for any weight in an irreducible representation are guaranteed to be integers by the MWDF, Eq. (5.1.23). Dynkin coefficients are independent of the normalization of simple roots, simple coroots, or fundamental weights; the same is true of the Cartan matrix $A_{ij} = \langle \overline{\alpha}_i, \overline{\alpha}_j^\vee \rangle = 2\langle \overline{\alpha}_i, \overline{\alpha}_j \rangle / \langle \overline{\alpha}_j, \overline{\alpha}_j \rangle$, the rows of which give the Dynkin coefficients for the simple roots.

By contrast, the quadratic form matrix $F_{ij} = \langle \omega_i, \omega_j \rangle$ [Eq. (8.1.5c)] and the quadratic Casimir eigenvalue in Eq. (9.1.10) do depend upon the normalization scheme. In certain applications (notably affine Lie algebras in conformal field theory (CFT) [2]), it is crucial to adopt a consistent normalization. We define the "standard" normalization scheme used in CFT and elsewhere in this subsection, and will employ it throughout the remainder of these lectures (unless otherwise noted).

Recall that all roots (including simple roots) come in one or two varieties for semi-simple Lie algebras: simply-laced algebras have all roots with the same length; otherwise there are "short" and "long" roots (Prop. IV., Sec. 5.2.2). The inner products of simple roots were computed for the four classical families in module 7, using a particular normalization scheme for the generators of the defining representation in each case. The results for $A_{n-1} = \mathrm{su}(n)$ [Eq. (7.1.21)], $B_n = \mathrm{so}(2n+1)$ [Eq. (7.4.19)], $C_n = \mathrm{sp}(2n)$ [Eq. (7.2.20)], and $D_n = \mathrm{so}(2n)$ [Eq. (7.3.23)] show that each long simple root $\overline{\alpha}_i^{(\mathsf{long})}$ for algebras B_n and C_n (all simple roots for simply-laced algebras, A_n and D_n) is normalized to

$$\langle \overline{\alpha}_i^{(\mathsf{long})}, \overline{\alpha}_i^{(\mathsf{long})} \rangle = |\overline{\alpha}_i^{(\mathsf{long})}|^2 = \frac{1}{g}, \tag{9.1.15}$$

where g is the **dual Coxeter number** defined in module 8, Eq. (8.1.13). The tabulated values for g in each case appear in Eq. (8.6.1), and match Eqs. (7.1.21), (7.4.19), (7.2.20), and (7.3.23).

The standard normalization scheme resets $|\overline{\alpha}_i^{(\mathsf{long})}|^2 = 2$ for all algebras, i.e. all long roots normalized to two. The highest root θ (Sec. 8.2.4) is always a long root, so that this scheme always has $|\theta|^2 = 2$. To achieve this, we renormalize the following objects:

- Renormalized Killing form:

$$(x, y)_2 \equiv \frac{1}{2g}(x, y) = \frac{1}{2g}\mathsf{Tr}\left[\widehat{ad}_x\, \widehat{ad}_y\right]_\theta. \tag{9.1.16a}$$

  The trace in the adjoint representation for a product of generators is $2g$ times the renormalized Killing form.

- Renormalized root representation in H, the Cartan subalgebra: Let $h_\alpha \in H$ represent the root $\alpha \in \Delta$; Δ denotes the set of roots. h_α is explicitly defined in the old normalization scheme via Eq. (4.4.10). Then we define the renormalized version

$$h_\alpha^{(2)} \equiv 2g\, h_\alpha \qquad \Rightarrow \qquad (h_\alpha^{(2)}, k)_2 = (h_\alpha, k) = \alpha(k). \tag{9.1.16b}$$

  Thus the eigenvalue of a root α acting on a generic element $k \in H$ is unchanged.

- Root scalar product: Let α and β belong to the set of roots Δ. Then we have the renormalized root scalar product

$$\langle \alpha, \beta \rangle_2 \equiv \left(h_\alpha^{(2)}, h_\beta^{(2)} \right)_2 = 2g\, \langle \alpha, \beta \rangle. \tag{9.1.16c}$$

Comparing Eqs. (9.1.15) and (9.1.16c) guarantees that

$$\langle \overline{\alpha}_i^{(\mathsf{long})}, \overline{\alpha}_i^{(\mathsf{long})} \rangle_2 = 2, \tag{9.1.17}$$

as desired. We also define a renormalized Casimir operator [c.f. Eq. (9.1.3a)],

$$\hat{C}_2 \equiv \sum_{i,j=1}^{n} \mathcal{A}_{2\,ij}^{-1} \hat{H}_{\overline{\alpha}_i}^{(2)}\, \hat{H}_{\overline{\alpha}_j}^{(2)} + \sum_{\alpha \neq 0} \frac{\hat{E}_\alpha\, \hat{E}_{-\alpha}}{(e_\alpha, e_{-\alpha})_2}, \qquad \text{Quadratic Casimir operator (standard normalization),} \tag{9.1.18a}$$

$$\mathcal{A}_{2\,ij} \equiv \langle \overline{\alpha}_i, \overline{\alpha}_j \rangle_2 = A_{ij}\, \frac{\langle \overline{\alpha}_j, \overline{\alpha}_j \rangle_2}{2}. \tag{9.1.18b}$$

It is evident that

$$\hat{C}_2 = 2g\, \hat{C}. \tag{9.1.19}$$

Now, the analog of the trace in Eqs. (9.1.5) and (9.1.6) is

$$(C_2)_2 \equiv \frac{1}{2g}\mathsf{Tr}\left[\hat{C}_2\right]_\theta = (C) = N_\theta. \tag{9.1.20}$$

Eqs. (9.1.10) and (9.1.20) then imply that

$$C_2^\theta = \langle\theta, \theta + 2\rho_{\mathsf{W}}\rangle_2 = 2g. \qquad \text{Standard Casimir in the adjoint representation} \\ = \text{twice the dual Coxeter number.} \tag{9.1.21}$$

If we express θ in terms of its comarks [Eq. (8.1.12)],

$$\theta = \sum_{j=1}^{n} a_j^\vee \,\overline{\alpha}_j^\vee, \tag{9.1.22}$$

then using $\langle\omega_i, \overline{\alpha}_j^\vee\rangle = \langle\omega_i, \overline{\alpha}_j^\vee\rangle_2 = \delta_{ij}$, we get

$$\langle\theta, \rho_{\mathsf{W}}\rangle_2 = \sum_{j=1}^{n} a_j^\vee. \tag{9.1.23}$$

Here we have used the fact that the Weyl vector ρ_{W} always has Dynkin coefficients $(1, 1, 1, \ldots, 1)$ [Eq. (9.1.9)]. Combined with the fact that $\langle\theta, \theta\rangle_2 = 2$, Eqs. (9.1.21) and (9.1.23) give

$$g = 1 + \sum_{j=1}^{n} a_j^\vee, \tag{9.1.24}$$

which is the formula for the dual Coxeter number given in terms of comarks by Eq. (8.1.13).

We can compute Eq. (9.1.21) explicitly, using (a) the Dynkin coefficients of the highest root $\theta = (\Lambda_1^{(\theta)}, \Lambda_2^{(\theta)}, \ldots, \Lambda_n^{(\theta)})$ (Sec. 8.2.4), (b) the Dynkin coefficients of the Weyl vector $\rho_{\mathsf{W}} = (1, 1, \ldots, 1)$ [Eq. (9.1.9)], and (c) the quadratic form matrix F_{ij}, already tabulated in the standard normalization for each algebra in Eq. (8.6.2). The Casimir for the adjoint representation is then

$$C_2^\theta = 2 + 2\sum_{i,j=1}^{n} \Lambda_i^{(\theta)} F_{ij}. \tag{9.1.25a}$$

For su$(n+1)$, $\Lambda_1^{(\theta)} = \Lambda_n^{(\theta)} = 1$ and all others vanish [Eq. (8.2.25)], while F_{ij} has the form in Eq. (8.6.2a). This leads to

$$C_2^{\theta\,(A_n)} = 2 + 2\sum_{j=1}^{n} (F_{1j} + F_{nj}) = 2 + \frac{2}{n+1}\,[n(n+1)] = 2(n+1). \tag{9.1.25b}$$

For sp$(2n)$, $\Lambda_1^{(\theta)} = 2$ and all others vanish [Eq. (8.2.27)], while F_{ij} has the form in Eq. (8.6.2c):

$$C_2^{\theta\,(C_n)} = 2 + 2^2\sum_{j=1}^{n} F_{1j} = 2(n+1). \tag{9.1.25c}$$

For so$(2n)$, $\Lambda_2^{(\theta)} = 1$ and all others vanish [Eq. (8.2.26)], while F_{ij} has the form in Eq. (8.6.2d):

$$C_2^{\theta\,(D_n)} = 2 + 2\sum_{j=1}^{n} F_{2j} = 2 + [3\times 2 + (n-3)4] = 2(2n-2). \tag{9.1.25d}$$

Eqs. (9.1.25b)–(9.1.25d) agree with the previously quoted results for $2g$ in Eq. (8.6.1).

For a generic representation with highest weight $\Lambda = (\Lambda_1, \cdots, \Lambda_n)$, the standard Casimir is

$$C_2^\Lambda = \langle\Lambda, \Lambda + 2\rho_{\mathsf{W}}\rangle_2 = \langle\Lambda, \Lambda\rangle_2 + 2\sum_{i,j=1}^{n} \Lambda_i\, F_{i,j}, \qquad \text{Standard Casimir for highest weight representation } \Lambda. \tag{9.1.26}$$

For the defining representation $\Lambda = \omega_1$, one finds the following

Algebra	Casimir $\langle \omega_1, \omega_1 + 2\rho_{\text{W}} \rangle_2$
$A_n = \text{su}(n+1)$	$\frac{n(n+2)}{n+1}$
$B_n = \text{so}(2n+1)$	$2n$
$C_n = \text{sp}(2n)$	$n + \frac{1}{2}$
$D_n = \text{so}(2n)$	$2n - 1$

(9.1.27)

• **Exercise**: Show that the standard Casimir for a rank-q fully antisymmetric tensor representation of A_n is given by

$$\langle \Lambda, \Lambda + 2\rho_{\text{W}} \rangle_2 = \frac{q(n+1-q)(n+2)}{n+1}, \quad 1 \leq q \leq n. \tag{9.1.28}$$

• **Exercise**: Show that the standard Casimir for a rank-q fully antisymmetric tensor representation of C_n is given by

$$\langle \Lambda, \Lambda + 2\rho_{\text{W}} \rangle_2 = \frac{1}{2} q \left[2(n+1) - q \right], \quad 1 \leq q \leq n. \tag{9.1.29}$$

9.1.2 Index of a representation; some examples

Let x and y be elements of the Lie algebra, $x, y \in L$. For a highest weight representation Λ, define

$$((x,y))_\Lambda \equiv \mathsf{Tr} \left[\hat{X} \, \hat{Y} \right]_\Lambda, \tag{9.1.30}$$

where the right-hand side is the trace normalization of the product of generators in the irreducible representation. The **index** l_Λ for representation Λ is then defined via

$$((x,y))_\Lambda = l_\Lambda \, (x,y)_2. \qquad \text{Index of a representation.} \tag{9.1.31}$$

The index relates the trace normalization in some arbitrary irreducible representation Λ to that in the adjoint representation (i.e., the renormalized Killing form). Using the renormalized Casimir operator constructed in the representation Λ, Eqs. (9.1.18a), (9.1.26), and (9.1.31) imply that

$$\begin{aligned} ((C_2)) &= l_\Lambda (C_2)_2 = l_\Lambda \, N_\theta \\ &= \langle \Lambda, \Lambda + 2\rho_{\text{W}} \rangle_2 \, N_\Lambda, \end{aligned} \tag{9.1.32}$$

which leads to

$$l_\Lambda = \frac{\langle \Lambda, \Lambda + 2\rho_{\text{W}} \rangle_2 \, N_\Lambda}{N_\theta}. \tag{9.1.33}$$

Here N_Λ is the dimension of the representation Λ (i.e., the total number of states, including degeneracies); N_θ is the dimension of the adjoint representation. Note that

$$l_\theta = \langle \theta, \theta + 2\rho_{\text{W}} \rangle = 2g, \tag{9.1.34}$$

consistent with Eq. (9.1.16a). The index of the defining representation l_{ω_1}, the adjoint representation $l_\theta = 2g$ [Eq. (8.6.1)], as well as the dimension N_{ω_1} and the dimension of the adjoint N_θ for the classical families are [via Eq. (9.1.27)]

Algebra	N_{ω_1}	N_θ	l_{ω_1}	$l_\theta = 2g$
$A_n = \text{su}(n+1)$	$n+1$	$n^2 + 2n$	1	$2(n+1)$
$B_n = \text{so}(2n+1)$	$2n+1$	$n(2n+1)$	2	$2(2n-1)$
$C_n = \text{sp}(2n)$	$2n$	$(2n)(n+1/2)$	1	$2(n+1)$
$D_n = \text{so}(2n)$	$2n$	$n(2n-1)$	2	$4(n-1)$

(9.1.35)

For the orthogonal algebras, it is interesting to compute the Casimirs and indices for the spinor representations.
• **Exercise**: Show that the standard Casimir for the ω_n fundamental spinor representation of $B_n = \text{so}(2n+1)$ is given by

$$\langle \omega_n, \omega_n + 2\rho_{\text{W}} \rangle_2 = \frac{n}{4}(2n+1). \tag{9.1.36}$$

• **Exercise**: Show that the standard Casimir for the ω_{n-1} and ω_n fundamental spinor representations of $D_n = \mathrm{so}(2n)$ is given by

$$\langle \omega_{n-1}, \omega_{n-1} + 2\rho_{\mathrm{W}} \rangle_2 = \langle \omega_n, \omega_n + 2\rho_{\mathrm{W}} \rangle_2 = \frac{n}{4}(2n-1). \tag{9.1.37}$$

Using the facts (proven later) that $N_{\omega_n} = 2^n$ in B_n and $N_{\omega_{n-1}} = N_{\omega_n} = 2^{n-1}$ for D_n, the indices are given by

Algebra	Fundamental spinor rep. index(ices)
$B_n = \mathrm{so}(2n+1)$	$l_{\omega_n} = 2^{n-2}$
$D_n = \mathrm{so}(2n)$	$l_{\omega_{n-1}} = l_{\omega_n} = 2^{n-3}$

(9.1.38)

9.1.2.1 su(2)

In su(2), the Cartan matrix is a single number: $A = \langle \theta, \theta \rangle = 2$. For a simply-laced algebra, $\alpha^\vee = 2\alpha/|\alpha|^2 = \alpha$ in the standard normalization scheme. Then the quadratic form matrix is $F = \langle \omega_1, \omega_1 \rangle = A^{-1} = 1/2$. The Weyl vector is $\rho_{\mathrm{W}} = \omega_1$. The spin j representation has highest weight $\Lambda = 2j$. The Casimir for this representation is

$$C_2^{(2j)} = 2j(2j+2)\langle \omega_1, \omega_1 \rangle_2 = 2j(j+1). \tag{9.1.39}$$

This is twice the usual normalization in quantum mechanics. The index is

$$l_{2j} = 2j(j+1)\frac{2j+1}{3} = \frac{2j(j+1)(2j+1)}{3}. \tag{9.1.40}$$

The index of the first few j values is

j	l_{2j}
0	0
1/2	1
1	4
3/2	10
2	20
5/2	35

(9.1.41)

Eq. (9.1.41) implies that the su(2) generators in the defining $j = 1/2$ representation are

$$\hat{T}_{x,y,z} = \frac{1}{\sqrt{2}}\hat{\sigma}^{x,y,z}, \qquad \left[\hat{T}_i, \hat{T}_j\right] = i f^{ijk}\hat{T}_k. \tag{9.1.42}$$

where $\hat{\sigma}^{x,y,z}$ are the standard Pauli matrices. The structure constants are $f^{ijk} = \sqrt{2}\epsilon^{ijk}$, which explains the "extra" factor of two in Eq. (9.1.39).

9.1.2.2 su(3)

A special feature of su(3) is that any highest weight representation can be associated to a traceless bisymmetrized tensor $\tilde{T}^{(j_1 \cdots j_q)}_{(i_1 \cdots i_p)}$, $\tilde{T}^{(k j_2 \cdots j_q)}_{(k i_2 \cdots i_p)} = 0$. The Dynkin coefficients of the associated highest weight are (p, q) [c.f. Eq. (8.5.2)]. The dimension is the number of independent elements of this tensor,

$$N_{p,q} = \frac{(p+2)!}{p!2!}\frac{(q+2)!}{q!2!} - \frac{(p+1)!}{(p-1)!2!}\frac{(q+1)!}{(q-1)!2!} = \frac{(p+1)(q+1)(p+q+2)}{2}. \tag{9.1.43}$$

This result can also be obtained from the "factors over hooks" rule. The Casimir for this representation is

$$\begin{aligned} C_2^{(p,q)} &= \langle p\,\omega_1 + q\,\omega_2, (p+2)\omega_1 + (q+2)\omega_2 \rangle \\ &= \frac{2}{3}\left[p(p+2) + q(q+2)\right] + \frac{1}{3}\left[p(q+2) + q(p+2)\right] = \frac{2}{3}\left[p^2 + q^2 + pq + 3(p+q)\right]. \end{aligned} \tag{9.1.44}$$

Thus the index is

$$l_{p,q} = \frac{C_2^{(p,q)} N_{p,q}}{N_\theta} = \frac{1}{24}(p+1)(q+1)(p+q+2)\left[p^2 + q^2 + pq + 3(p+q)\right]. \tag{9.1.45}$$

The Casimir and index for a few low-dimensional representations are

HWS	conjugacy class	$N_{p,q}$	$C_2^{(p,q)}$	$l_{p,q}$
$(0,0)$	0	1	0	0
$(1,0)$	1	3	8/3	1
$(1,1)$	0	8	6	6
$(2,0)$	2	6	20/3	5
$(2,1)$	1	15	32/3	20
$(2,2)$	0	27	16	54

(9.1.46)

Eqs. (9.1.35), (9.1.38), (9.1.41), and (9.1.46) suggest that the index is always a non-negative integer; this is indeed the case.

9.2 Freudenthal's recursion formula for weight space dimensions

The master weight depth formula Eq. (5.1.23) allows the determination of all weights in a finite-dimensional, highest weight irreducible representation of a Lie algebra. It does *not* determine the total number of states in a given representation, however, because some weights can be degenerate. For example, the (2,1) representation of su(3) shown in Fig. 9.1 has only 12 distinct weights. On the other hand, Eq. (9.1.43) implies that $N_{2,1} = 15$. Therefore between one to three of the weights in (2,1) must be degenerate. Recall that the **weight space** refers to the set of states sharing the same weight, i.e. the same Dynkin coefficients.

In this section we will determine a recursive algorithm (Freudenthal's formula) to determine the dimension of all weight spaces in a given representation. Let M be a weight in the representation with highest weight Λ. Consider the trace of the Casimir [Eq. (9.1.26)] within the possibly degenerate weight space associated to M,

$$\begin{aligned}\mathsf{Tr}\left[\hat{C}_2^{\Lambda}\right]_M &= n_M\,\langle\Lambda,\Lambda+2\rho_{\mathsf{W}}\rangle_2 \\ &= n_M\langle M,M\rangle_2 + \sum_{\alpha>0}\frac{\mathsf{Tr}_M\left[\hat{E}_\alpha\,\hat{E}_{-\alpha}+\hat{E}_{-\alpha}\,\hat{E}_\alpha\right]}{(e_\alpha,e_{-\alpha})_2},\end{aligned} \tag{9.2.1}$$

where n_M is the dimension of the weight space labeled by M. In order to evaluate the second term, we consider the su(2) generated by

$$\begin{aligned}[\hat{H}_\alpha^{(2)},\hat{E}_{\pm\alpha}] &= \pm\,\langle\alpha,\alpha\rangle_2\,\hat{E}_{\pm\alpha}, \\ [\hat{E}_\alpha,\hat{E}_{-\alpha}] &= (e_\alpha,e_{-\alpha})_2\,\hat{H}_\alpha^{(2)}.\end{aligned} \tag{9.2.2}$$

It is useful to rewrite this in terms of standard su(2) generators,

$$[\hat{T}_z,\hat{T}_\pm] = \pm\hat{T}_\pm, \qquad [\hat{T}_+,\hat{T}_-] = 2\hat{T}_z. \tag{9.2.3}$$

Let

$$\hat{T}_\pm \equiv A\,\hat{E}_{\pm\alpha}, \quad \hat{T}_z \equiv B\,\hat{H}_\alpha, \quad A = \sqrt{\frac{2}{\langle\alpha,\alpha\rangle_2(e_\alpha,e_{-\alpha})_2}}, \quad B = \frac{1}{\langle\alpha,\alpha\rangle_2}. \tag{9.2.4}$$

For the particular root α, we have the su(2) Casimir in Eq. (9.1.1),

$$\frac{1}{2}\left(\hat{T}_+\hat{T}_- + \hat{T}_-\hat{T}_+\right) + \hat{T}_z\hat{T}_z = \frac{A^2}{2}\left[\hat{E}_\alpha\,\hat{E}_{-\alpha}+\hat{E}_{-\alpha}\,\hat{E}_\alpha\right] + B^2\,\hat{H}_\alpha^2. \tag{9.2.5}$$

Now, each state in the n_M-fold degenerate weight space of M belongs to some irreducible representation of this su(2) with highest weight $t \in \mathbb{Z}/2$, Casimir eigenvalue $t(t+1)$. We choose a basis within the weight space such that each weight vector belongs to a distinct α-generated su(2) representation. The weight space may contain multiple distinguishable copies of the same irreducible representation and associated highest weight t. It may also possess one or more isolated states, i.e. states with weight M that cannot be raised or lowered by $\hat{E}_{\pm\alpha} = \hat{T}_\pm/A$.

Consider the state in the weight space of M that sits in the α-generated su(2) with highest weight t. Denote this state by the weight vector $\phi_t^{(M)}$. Suppose that $\phi_t^{(M+K\alpha)}$ is the associated highest weight:

$$\hat{T}_z\,\phi_t^{(M+K\alpha)} = t\,\phi_t^{(M+K\alpha)} = B\langle\alpha,M+K\alpha\rangle_2\,\phi_t^{(M+K\alpha)} \quad\Rightarrow\quad t = B\langle\alpha,M+K\alpha\rangle_2. \tag{9.2.6}$$

Then

$$\left[\hat{E}_\alpha\,\hat{E}_{-\alpha}+\hat{E}_{-\alpha}\,\hat{E}_\alpha\right]\phi_t^{(M)}=\left\{-\frac{2B^2}{A^2}\,\langle M,\alpha\rangle_2^2+\frac{2B}{A^2}\langle\alpha,M+K\alpha\rangle_2+\frac{2B^2}{A^2}\left[\langle\alpha,M\rangle_2^2+K^2\langle\alpha,\alpha\rangle_2^2+2K\langle M,\alpha\rangle_2\langle\alpha,\alpha\rangle_2\right]\right\}\phi_t^{(M)}$$
$$=\frac{2B}{A^2}\left[(2K+1)\,\langle M,\alpha\rangle_2+K(K+1)\,\langle\alpha,\alpha\rangle_2\right]\phi_t^{(M)}. \tag{9.2.7}$$

Next we note that the number of α-generated su(2) representations with label K marking the highest weight is

$$n_{M+\alpha K}-n_{M+\alpha(K+1)}. \tag{9.2.8}$$

The first term is the degeneracy of the weight space for the presumed highest weight. The second term is the degeneracy of the weight space one step above the highest weight; taking the difference eliminates the contributions of degenerate states with weight M that originate by lowering from a *higher* highest weight state than that assumed. To get the total contribution from the weight space trace on the right-hand side of Eq. (9.2.1), we sum over all possible values of K. I.e., we sum over all possible highest weight states that can give M as a descendant:

$$\sum_{\alpha>0}\frac{\mathsf{Tr}_M\left[\hat{E}_\alpha\,\hat{E}_{-\alpha}+\hat{E}_{-\alpha}\,\hat{E}_\alpha\right]}{(e_\alpha,e_{-\alpha})_2}=\sum_{\alpha>0}\sum_{K\geq 0}\left[n_{M+\alpha K}-n_{M+\alpha(K+1)}\right]\frac{2B}{A^2(e_\alpha,e_{-\alpha})_2}\left[(2K+1)\,\langle M,\alpha\rangle_2+K(K+1)\,\langle\alpha,\alpha\rangle_2\right]$$
$$=\sum_{\alpha>0}\sum_{K\geq 0}\left[n_{M+\alpha K}-n_{M+\alpha(K+1)}\right]\left[(2K+1)\,\langle M,\alpha\rangle_2+K(K+1)\,\langle\alpha,\alpha\rangle_2\right]$$
$$=\sum_{\alpha>0}\left\{n_M\langle M,\alpha\rangle_2+\sum_{K\geq 1}n_{M+\alpha K}\left[2\,\langle M,\alpha\rangle_2+2K\,\langle\alpha,\alpha\rangle_2\right]\right\}$$
$$=\left\{n_M\langle M,2\rho_{\mathsf{W}}\rangle_2+\sum_{\alpha>0}\sum_{K\geq 1}2n_{M+\alpha K}\left[\langle M,\alpha\rangle_2+K\,\langle\alpha,\alpha\rangle_2\right]\right\}$$
$$=n_M\left[\langle\Lambda,\Lambda+2\rho_{\mathsf{W}}\rangle_2-\langle M,M\rangle_2\right]. \tag{9.2.9}$$

Solving for n_M leads to

$$n_M=\frac{\displaystyle\sum_{\alpha>0}\sum_{K\geq 1}2\,n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle_2}{\langle\Lambda+M+2\rho_{\mathsf{W}},\Lambda-M\rangle_2}, \tag{9.2.10}$$

Freudenthal's recursion formula for the dimension of weight space M.

An algorithm to evaluate Eq. (9.2.10) is the following:

- Start with the highest weight of the representation Λ. This is guaranteed to be non-degenerate, so $n_\Lambda=1$.
- Proceed to each weight that differs from Λ by one root. To compute the inner products in the numerator and denominator of Eq. (9.2.10), one can express all weights in terms of Dynkin coefficients and use the quadratic form matrix [Eq. (8.6.2)]. Alternatively, to compute $\langle\Lambda,\Lambda'\rangle$ one can express

$$\Lambda=\sum_{i=1}^{n}\Lambda_i\omega_i,\quad \Lambda'=\sum_{j=1}^{n}\kappa_j^{(\Lambda')}\,\overline{\alpha}_j, \tag{9.2.11}$$

so that

$$\langle\Lambda,\Lambda'\rangle=\sum_{i=1}^{n}\Lambda_i\,\kappa_i^{(\Lambda')}\,\frac{|\overline{\alpha}_i|_2^2}{2}. \tag{9.2.12}$$

Here we have exploited the orthonormality of the fundamental weights and simple coroots; the factor of $|\overline{\alpha}_i|^2/2$ appears because we have expanded Λ' in terms of simple roots. Note that for a simply-laced algebra, $|\overline{\alpha}_i|_2^2=2$.

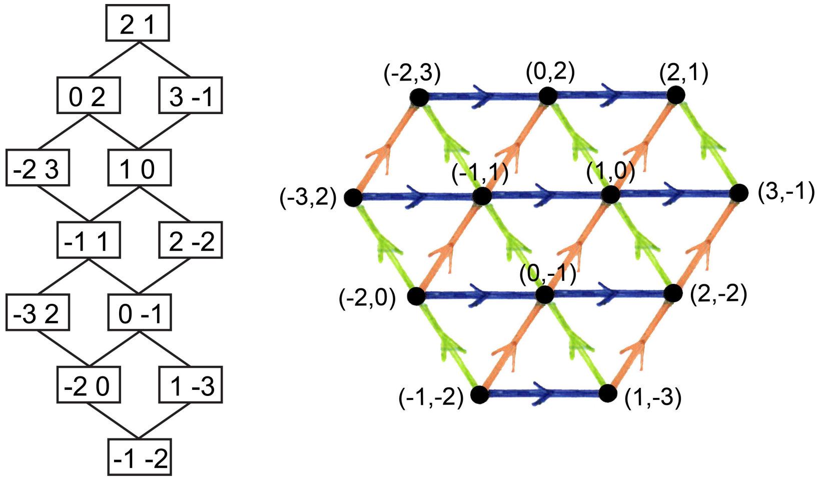

Figure 9.1: (2,1) representation of su(3). There are 15 states in this representation, but only 12 distinct weights. Each of the weights in the inner triangle are doubly degenerate.

We will illustrate Eq. (9.2.10) for the (2,1) representation of the simply-laced algebra su(3), Fig. 9.1.

1. The highest weight state has $n_{2,1} = 1$.
2. Consider $M = (0,2)$. It has

$$\begin{aligned} \Lambda + M + 2\rho_{\rm w} = (2,1) + (0,2) + (2,2) = (4,5), \qquad \Lambda - M = \overline{\alpha}_1, \\ \Rightarrow \quad \langle \Lambda + M + 2\rho_{\rm w}, \Lambda - M \rangle_2 = 4, \\ K = 1: \quad 2\, n_{M+\overline{\alpha}_1} \, \langle M + \overline{\alpha}_1, \overline{\alpha}_1 \rangle_2 = 2\,(0+2) = 4, \end{aligned} \tag{9.2.13}$$

so that $n_{0,2} = 1$.
3. $M = (-2,3)$.

$$\begin{aligned} \Lambda + M + 2\rho_{\rm w} = (2,1) + (-2,3) + (2,2) = (2,6), \qquad \Lambda - M = 2\overline{\alpha}_1, \\ \Rightarrow \quad \langle \Lambda + M + 2\rho_{\rm w}, \Lambda - M \rangle_2 = 4, \\ K = 1: \quad 2\, n_{M+\overline{\alpha}_1} \, \langle M + \overline{\alpha}_1, \overline{\alpha}_1 \rangle_2 = 0, \\ K = 2: \quad 2\, n_{M+2\overline{\alpha}_1} \, \langle M + 2\overline{\alpha}_1, \overline{\alpha}_1 \rangle_2 = 2\,(-2+4) = 4, \end{aligned} \tag{9.2.14}$$

so that $n_{-2,3} = 1$.
4. $M = (3,-1)$.

$$\begin{aligned} \Lambda + M + 2\rho_{\rm w} = (2,1) + (3,-1) + (2,2) = (7,2), \qquad \Lambda - M = \overline{\alpha}_2, \\ \Rightarrow \quad \langle \Lambda + M + 2\rho_{\rm w}, \Lambda - M \rangle_2 = 2, \\ K = 1: \quad 2\, n_{M+\overline{\alpha}_2} \, \langle M + \overline{\alpha}_2, \overline{\alpha}_2 \rangle_2 = 2\,(-1+2) = 2, \end{aligned} \tag{9.2.15}$$

so that $n_{3,-1} = 1$.
5. $M = (1,0)$.

$$\begin{aligned} \Lambda + M + 2\rho_{\rm w} = (2,1) + (1,0) + (2,2) = (5,3), \qquad \Lambda - M = \overline{\alpha}_1 + \overline{\alpha}_2, \\ \Rightarrow \quad \langle \Lambda + M + 2\rho_{\rm w}, \Lambda - M \rangle_2 = 8, \\ \overline{\alpha}_1, \quad K = 1: \quad 2\, n_{M+\overline{\alpha}_1} \, \langle M + \overline{\alpha}_1, \overline{\alpha}_1 \rangle_2 = 2(2+1) = 6, \\ \overline{\alpha}_2, \quad K = 1: \quad 2\, n_{M+\overline{\alpha}_2} \, \langle M + \overline{\alpha}_2, \overline{\alpha}_2 \rangle_2 = 2(2) = 4, \\ \overline{\alpha}_1 + \overline{\alpha}_2, \quad K = 1: \quad 2\, n_{M+\overline{\alpha}_1+\overline{\alpha}_2} \, \langle M + \overline{\alpha}_1 + \overline{\alpha}_2, \overline{\alpha}_1 + \overline{\alpha}_2 \rangle_2 = 2(3) = 6, \end{aligned} \tag{9.2.16}$$

so that $n_{1,0} = (6+4+6)/8 = 2$.

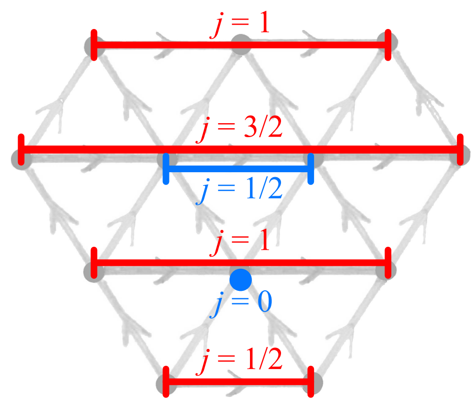

Figure 9.2: The same (2,1) representation of su(3) as shown in Fig. 9.1. Here we depict the decomposition into irreducible su(2) representations generated by $\hat{E}_{\pm\overline{\alpha}_1}$, i.e. the decomposition into parallel strings consisting of weights separated by simple root $\overline{\alpha}_1$ moves. The doubly-degenerate weights arise due to overlapping representations.

We could proceed to evaluate the remaining weight space dimensions, but it is much easier to simply infer them using Weyl reflections, see Fig. 9.3. We conclude that the weights (1,0), (-1,1), and (0,-1) are each doubly-degenerate, while each weight in the "outer layer" (see Fig. 9.1) has a one-dimensional weight space. This gives a total of 12 + 3 = 15 weights, in agreement with Eq. (9.1.43).

The partitioning of the (2,1) representation into different and sometimes overlapping su(2) representations generated by $\hat{E}_{\pm\overline{\alpha}_1}$ is depicted in Fig. 9.2.

• Exercise: Determine the weight tree and the dimension of all weight spaces for the su(3) representation (2,2), which has 27 total states [Eq. (9.1.46)]. Is there a pattern to the degeneracies in terms of the weight geometry?

• Exercise: Use Freudenthal's formula to show that all states of the fundamental spinor representation (0,0,1) of so(7) correspond to distinct weights, i.e. that all weight spaces are one-dimensional. The weight tree (geometry) is shown in Fig. 8.9 (Fig. 8.10).

9.3 Weyl group

• • • Throughout the remainder of the course, unless stated otherwise we will employ the "standard normalization" defined by Eq. (9.1.17). We will drop explicit "2" subscripts to lighten the notation.

The **Weyl reflection** S_α along a root α was introduced in Sec. 5.2.1, in the context of root system geometries. The action of S_α upon a generic weight M in an irreducible representation is given by [Eq. (5.2.3)]

$$S_\alpha\, M = M - \langle M, \alpha^\vee \rangle\, \alpha, \quad S_\alpha^2 = 1, \qquad \text{Weyl reflection } \equiv S_\alpha. \tag{9.3.1}$$

With respect to the su(2) generated by $\{\hat{E}_{\pm\alpha}, \hat{H}_\alpha\}$, an irreducible representation decomposes into parallel strings of weights of varying lengths. Applying the Weyl reflection interchanges the weights along these strings. Weyl reflections for su(3) are depicted in Fig. 9.3. A **simple Weyl reflection** $S_{\overline{\alpha}_i}$ gives

$$S_{\overline{\alpha}_i}\, M = M - \Lambda_i^{(M)}\, \overline{\alpha}_i, \qquad \text{Simple Weyl reflection,} \tag{9.3.2}$$

where $\Lambda_i^{(M)}$ is the corresponding Dynkin coefficient. Acting on another simple root gives

$$S_{\overline{\alpha}_i}\, \overline{\alpha}_j = \overline{\alpha}_j - A_{ji}\, \overline{\alpha}_i, \tag{9.3.3}$$

where $A_{ji} \equiv \langle \overline{\alpha}_j, \overline{\alpha}_i^\vee \rangle$ is the Cartan Matrix [Eq. (5.3.8)].

We prove the following facts about Weyl reflections.

1. Invariance of the scalar product.

$$\langle S_\alpha\, M, M' \rangle = \langle M - \langle M, \alpha^\vee \rangle \alpha, M' \rangle = \langle M, M' \rangle - \langle M, \alpha^\vee \rangle \langle M', \alpha \rangle = \langle M, S_\alpha\, M' \rangle. \tag{9.3.4}$$

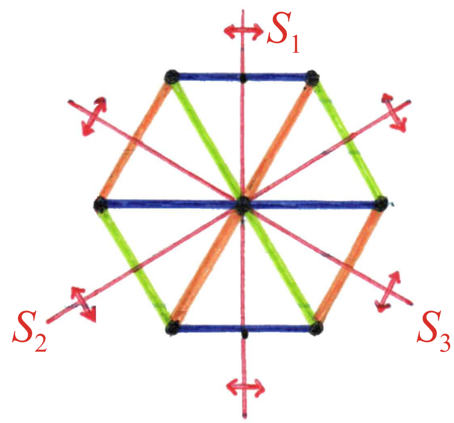

Figure 9.3: Weyl reflections $S_{1,2,3}$ for su(3).

2. The simple Weyl reflection $S_{\overline{\alpha}_i}$ permutes positive roots, except for $\overline{\alpha}_i \to -\overline{\alpha}_i$. Consider a positive root $\beta \in \Delta_+$ [Eq. (5.3.7)]

$$\beta \equiv \sum_{i=1}^{n} \kappa_i^{(\beta)} \overline{\alpha}_i, \quad \kappa_i^{(\beta)} \in \mathbb{N}_0. \tag{9.3.5}$$

The action of a simple root Weyl reflection is

$$S_{\overline{\alpha}_i}\, \beta = \sum_{j \neq i}^{n} \kappa_j^{(\beta)} \overline{\alpha}_j + \overline{\alpha}_i \left[\sum_{j \neq i}^{n} \kappa_j^{(\beta)} \left| A_{ji} \right| - \kappa_i^{(\beta)} \right]. \tag{9.3.6}$$

Now, assuming that the positive root $\beta \neq \overline{\alpha}_i$, then at least one $\kappa_{j \neq i}^{(\beta)} > 0$. But recall that in the "root-building" application of the MWDF [Eqs. (5.3.26) and (7.1.24)], each positive root obtains by adding some combination of simple roots to a particular simple root. It means that if one expansion coefficient in Eq. (9.3.6) is positive, then so are all others. Therefore $S_{\overline{\alpha}_i}\, \beta$ is a positive root.

Using Weyl reflections, we can prove the equivalence of the two expressions for the Weyl vector ρ_{W} in Eq. (9.1.9). Let

$$\sigma \equiv \frac{1}{2} \sum_{\alpha > 0} \alpha, \qquad \rho \equiv \sum_{j=1}^{n} \omega_j. \tag{9.3.7}$$

Since a simple Weyl reflection $S_{\overline{\alpha}_i}$ permutes the positive roots except $\overline{\alpha}_i$,

$$S_{\overline{\alpha}_i}\, \sigma = \frac{1}{2} \sum_{\substack{\alpha > 0 \\ \alpha \neq \overline{\alpha}_i}} \alpha - \frac{1}{2} \overline{\alpha}_i = \sigma - \overline{\alpha}_i. \tag{9.3.8}$$

Then

$$\langle S_{\overline{\alpha}_i}\, \sigma, \overline{\alpha}_i^{\vee} \rangle = \langle \sigma, \overline{\alpha}_i^{\vee} \rangle - 2 = \langle \sigma, S_{\overline{\alpha}_i}\, \overline{\alpha}_i^{\vee} \rangle = -\langle \sigma, \overline{\alpha}_i^{\vee} \rangle, \tag{9.3.9}$$

or

$$\langle \sigma, \overline{\alpha}_i^{\vee} \rangle = \Lambda_i = 1, \qquad i \in \{1, 2, \ldots, n\}. \tag{9.3.10}$$

In this equation $\{\Lambda_i\}$ denote the Dynkin coefficients of σ. We conclude that $\sigma = \rho = \rho_{\mathsf{W}}$ in Eq. (9.3.7).

The product of all Weyl reflections forms the **Weyl group** W, which is a subgroup of the isometry group for the root system. Any weight geometry associated to an irreducible representation must be left invariant under a Weyl group transformation. Note that the Weyl group contains more than Weyl reflections, since the product of two reflections is equivalent to a rotation. The product of an even (odd) even number of Weyl reflections is a "proper rotation" (an "improper rotation") with determinant equal to +1 (−1) (thinking of S_α as an orthogonal transformation on n-dimensional vectors in H_0^*, the space of distinct weights).

We can choose a "basis" for the Weyl group by considering compositions of simple root Weyl reflections $\{S_{\overline{\alpha}_j}\}$, $1 \le j \le n$ (and n denotes the rank of the algebra). Every element $w \in W$ can be written as a product of simple Weyl reflections,

$$w = S_{\overline{\alpha}_{i_1}} S_{\overline{\alpha}_{i_2}} \times \cdots \times S_{\overline{\alpha}_{i_p}}. \tag{9.3.11}$$

The Weyl group is important because it relates weights based on *point group symmetries* of the representation geometry. In module 8, we mainly stressed tools that do not make this geometry manifest. We focused on expanding weights in terms of the fundamental weights $\{\omega_i\}$ with Dynkin expansion coefficients, the MWDF algorithm for constructing the weight tree, Young tableaux, etc. The geometry of a representation is instead encoded in the expansion of a generic weight in terms of simple roots and the geometry of the root system. Weyl group transformations algebraically encode this geometrical information.

An obvious but important statement is the following. Given the set of simple roots, the entire root geometry can be constructed by acting on simple roots with elements of the Weyl group. In other words,

$$\Delta = \left\{ w\,\overline{\alpha}_1, w\,\overline{\alpha}_2, \ldots, w\,\overline{\alpha}_n \middle| w \in W \right\}. \tag{9.3.12}$$

Here Δ denotes the set of all roots. Evidently any set $\{w\,\overline{\alpha}_i\}$ for a particular, fixed Weyl element w could serve as a basis of simple roots. Thus we can use the Weyl group to change the basis for the simple roots.

Another obvious implication is that

$$n_M = n_{(w\,M)}, \quad w \in W. \tag{9.3.13}$$

Here n_M is the dimension of the weight space for weight M in some irreducible representation [as appears in Freudenthal's formula Eq. (9.2.10)] and w is a generic Weyl group element.

The set of weights that can be obtained by acting with all possible Weyl group operations on some particular weight M is called the **Weyl orbit** of M. We can choose one particular weight in this orbit with non-negative Dynkin coefficients to represent the entire orbit. **Proof:** Suppose that M' is the highest weight in the orbit, and suppose that it has a negative Dynkin coefficient: $\Lambda_i = \langle M', \overline{\alpha}_i^\vee \rangle < 0$. But then

$$S_{\overline{\alpha}_i} M' = M' - \langle M', \overline{\alpha}_i^\vee \rangle \overline{\alpha}_i = M' + |\Lambda_i|\, \overline{\alpha}_i \tag{9.3.14}$$

is by definition (lexicographic ordering, Sec. 5.3.1) greater than M' (contradiction). A weight with non-negative Dynkin coefficients is said to be **dominant**.

9.4 Weyl's character formula: derivation

9.4.1 Generating functions

In the following, we will need to view H_0^* (the space of roots with real coefficients) in two different ways:

1. As a linear vector space of dimension n (where n denotes the rank), with the linearly independent simple root basis. A generic element is a "vector" (weight)

$$\beta = \sum_{j=1}^{n} \kappa_j^{(\beta)} \overline{\alpha}_j, \qquad \langle \overline{\alpha}_i, \overline{\alpha}_j \rangle = \mathcal{A}_{ij} = A_{ij} \frac{|\overline{\alpha}_j|^2}{2}. \tag{9.4.1}$$

2. As an infinite dimensional Hilbert space, where $|\beta\rangle$ corresponding to a generic weight is itself a <u>basis element</u>. The inner product between two such states is

$$\langle \beta | \lambda \rangle = \delta^{(n)}(\beta - \lambda).$$

A generic state is a <u>linear function</u>,

$$|f\rangle = \int d^n\beta\, f(\beta)\, |\beta\rangle\,. \tag{9.4.2}$$

We can view the action of a Weyl group transformation $w \in W$ in these two different ways:

1. As an orthogonal transformation on an n-component vector,

$$w\,\beta = \sum_j \kappa_j^{(\beta)} (w\,\overline{\alpha}_j), \tag{9.4.3}$$

where $w\,\overline{\alpha}_j$ is the transformed basis vector.

2. As a unitary Hilbert space operator $\hat{U}_w$ such that

$$\hat{U}_w\,|\beta\rangle = |w\,\beta\rangle\,. \tag{9.4.4}$$

The action on a generic function is

$$\hat{U}_w\,|f\rangle = \int d^n\beta\, f(\beta)\,|w\,\beta\rangle = \int d^n\beta\, f(w^{-1}\beta)\,|\beta\rangle\,, \tag{9.4.5}$$

so that

$$(w\,f)\,(\beta) \equiv \langle\beta|\,\hat{U}_w\,|f\rangle = f(w^{-1}\beta). \tag{9.4.6}$$

Now we define two different functions on H_0^*:

$$\chi(\beta) = \langle\beta\,|\chi\rangle \equiv \sum_M n_M\, e^{\langle M,\beta\rangle}, \qquad \text{Character of a representation.} \tag{9.4.7a}$$

$$Q(\beta) = \langle\beta\,|Q\rangle \equiv \prod_{\alpha>0}\left[e^{\frac{1}{2}\langle\alpha,\beta\rangle} - e^{-\frac{1}{2}\langle\alpha,\beta\rangle}\right], \qquad \text{(Auxiliary) } Q\text{-function.} \tag{9.4.7b}$$

The character function $\chi(\beta)$ involves a sum over all weights $\{M\}$ of an irreducible representation; n_M is the weight space dimension [Eq. (9.2.10)]. The Q-function involves a product over the positive roots.

The character formula is invariant under a Weyl transformation:

$$(w\,\chi)\,(\beta) = \chi(w^{-1}\beta) = \sum_M n_M\, e^{\langle w\,M,\beta\rangle} = \sum_M n_{w^{-1}\,M}\, e^{\langle M,\beta\rangle} = \chi(\beta), \tag{9.4.8}$$

where we have used Eq. (9.3.13).

Let $\hat{U}_{\overline{\alpha}_i}$ implement a simple Weyl reflection [Eq. (9.3.2)],

$$\hat{U}_{\overline{\alpha}_i}\,|\beta\rangle = |S_{\overline{\alpha}_i}\beta\rangle\,. \tag{9.4.9}$$

Then

$$\begin{aligned}(S_{\overline{\alpha}_i}Q)\,(\beta) = \langle\beta|\,\hat{U}_{\overline{\alpha}_i}\,|Q\rangle &= \prod_{\alpha>0}\left[e^{\frac{1}{2}\langle S_{\overline{\alpha}_i}\alpha,\beta\rangle} - e^{-\frac{1}{2}\langle S_{\overline{\alpha}_i}\alpha,\beta\rangle}\right]\\ &= \prod_{\substack{\alpha>0\\ \alpha\neq\overline{\alpha}_i}}\left[e^{\frac{1}{2}\langle\alpha,\beta\rangle} - e^{-\frac{1}{2}\langle\alpha,\beta\rangle}\right]\left[e^{-\frac{1}{2}\langle\overline{\alpha}_i,\beta\rangle} - e^{\frac{1}{2}\langle\overline{\alpha}_i,\beta\rangle}\right] = -Q(\beta).\end{aligned} \tag{9.4.10}$$

Here we have used the fact that $S_{\overline{\alpha}_i}$ permutes the positive roots *except* for $\overline{\alpha}_i \to -\overline{\alpha}_i$. More generally, can show that

$$(w\,Q)\,(\beta) = Q(w^{-1}\beta) = (\det w)\;Q(\beta), \qquad w \in W. \tag{9.4.11}$$

Here $\det w$ is the determinant of the Weyl group element w, represented as an $n\times n$ matrix acting on the linear vector space H_0^* spanned by $\{\overline{\alpha}_i\}$. This determinant is guaranteed to be equal to plus or minus one for a proper or improper rotation, respectively. Eq. (9.4.11) implies that $Q(\beta)$ is an "alternating" function on weight space. Equivalently, $|Q\rangle$ is an eigenvector of a generic Weyl transformation $\hat{U}_w$ with eigenvalue $\det w$,

$$\hat{U}_w\,|Q\rangle = (\det w)\,|Q\rangle\,. \tag{9.4.12}$$

Define

$$\hat{U}_\sigma \equiv \sum_{w\in W}(\det w)\,\hat{U}_w, \qquad \text{"Alternating" (and un-normalized) projector.} \tag{9.4.13}$$

This has the properties

$$\hat{U}_w\,\hat{U}_\sigma = \hat{U}_\sigma\,\hat{U}_w = (\det w)\,\hat{U}_\sigma, \tag{9.4.14a}$$

$$(\sigma\,f)\,(\beta) \equiv \langle\beta|\,\hat{U}_\sigma\,|f\rangle = \sum_{w\in W}(\det w)\,f(w^{-1}\beta), \tag{9.4.14b}$$

$$\hat{U}_\sigma^2 = N_W\,\hat{U}_\sigma, \tag{9.4.14c}$$

where N_W is the order (number of elements) of the Weyl group.

Let us attempt a decomposition for $Q(\beta)$ using $\hat{U}_\sigma$:

$$Q(\beta) \equiv (\sigma\, R)\,(\beta), \tag{9.4.15}$$

where $R(\beta)$ is some other function. Eq. (9.4.14a) guarantees that $Q(\beta)$ constructed in this way is alternating. If we expand the product in Eq. (9.4.7b), we will get terms of the form $\exp\left[\left\langle\sum_{\alpha>0}(\pm 1)^{m_\alpha}\,\frac{\alpha}{2},\beta\right\rangle\right]$, where $m_\alpha\in\{0,1\}$. The term with all $m_\alpha=0$ or all $m_\alpha=1$ is $\exp\left(\langle\pm\rho_{\mathsf{W}},\beta\rangle\right)$ [Eq. (9.1.9)]. We therefore assume an expansion of the form

$$R(\beta) \equiv c_0\, f_{\rho_{\mathsf{W}}}(\beta) + \sum_{\alpha>0} c_\alpha\, f_{\rho_{\mathsf{W}}-\alpha}(\beta), \qquad f_M(\beta) \equiv \exp\left[\langle M,\beta\rangle\right]. \tag{9.4.16}$$

Note that

$$(\sigma\, f_M)\,(\beta) = \sum_{w\in W} (\det w)\, \exp\left[\langle wM,\beta\rangle\right] \tag{9.4.17a}$$

$$(\sigma\, f_{w\,M})\,(\beta) = (\det w)\;(\sigma\, f_M)\,(\beta). \tag{9.4.17b}$$

Eq. (9.4.17b) implies that there is no point including different weights related by Weyl transformations in the expansion Eq. (9.4.16), since their contributions to $Q(\beta)$ are equivalent up to a sign.

We can therefore restrict the sum in Eq. (9.4.16) to weights $M=\rho_{\mathsf{W}}-\alpha$ with non-negative Dynkin coefficients [c.f. Eq. (9.3.14) and the surrounding discussion]. In fact, $(\sigma\, f_M)\,(\beta)=0$ for any weight M with at least one vanishing Dynkin label.

Proof:

- Suppose the Dynkin coefficient $\Lambda_i^{(M)}=0$. Then [Eq. (9.3.2)]

$$S_{\overline{\alpha}_i} M = M.$$

- The function $f_M(\beta)$ is invariant,

$$(S_{\overline{\alpha}_i}\, f_M)\,(\beta) = f_M\,(S_{\overline{\alpha}_i}\,\beta) = f_{S_{\overline{\alpha}_i} M}(\beta) = f_M(\beta) \quad\Rightarrow\quad \left(\sigma\, f_{S_{\overline{\alpha}_i} M}\right)(\beta) = (\sigma\, f_M)\,(\beta).$$

- But then Eq. (9.4.17b) implies that

$$(\sigma\, S_{\overline{\alpha}_i}\, f_M)\,(\beta) = \left(\sigma\, f_{S_{\overline{\alpha}_i} M}\right)(\beta) = -\,(\sigma\, f_M)\,(\beta).$$

  Therefore $(\sigma\, f_M)\,(\beta)=0$.

Now, the Dynkin coefficients of the Weyl vector ρ_{W} are $(1,1,\ldots,1)$; therefore, any weight of the form $M=\rho_{\mathsf{W}}-\alpha$ is guaranteed to have a negative or zero Dynkin coefficient. We conclude that $R(\beta)$ in Eq. (9.4.16) reduces to

$$R(\beta) = c_0\,\exp\left[\langle\rho_{\mathsf{W}},\beta\rangle\right]. \tag{9.4.18}$$

Then

$$Q(\beta) = c_0 \sum_{w\in W} (\det w)\, e^{\langle w\,\rho_{\mathsf{W}},\beta\rangle} = c_0\, e^{\langle\rho_{\mathsf{W}},\beta\rangle} + (\text{terms involving Weyl transformations of } \rho_{\mathsf{W}}). \tag{9.4.19}$$

If we compare this to Eq. (9.4.7b), we conclude that $c_0=1$; therefore

$$Q(\beta) = (\sigma\, f_{\rho_{\mathsf{W}}})\,(\beta) = \sum_{w\in W} (\det w)\, e^{\langle w\,\rho_{\mathsf{W}},\beta\rangle} = \sum_{w\in W} (\det w)\, e^{\langle\rho_{\mathsf{W}},w\,\beta\rangle}. \tag{9.4.20}$$

The last equality holds because $\det w^{-1}=\det w$ for an orthogonal Weyl group transformation w.

9.4.2 Freudenthal redux

Freudenthal's formula [Eq. (9.2.10)] can be rewritten as

$$[\langle\Lambda,\Lambda\rangle - \langle M,M\rangle + \langle 2\rho_{\rm w},\Lambda - M\rangle]\, n_M = 2\sum_{\alpha>0}\sum_{K\geq 1} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle. \tag{9.4.21}$$

It is useful to recast this in slightly different form.
Claim:

$$\sum_{K=-\infty}^{\infty} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle = 0. \tag{9.4.22}$$

In this equation, it is understood that $n_{M+\alpha K}=0$ for weights in the chain generated by $\hat{E}_{\pm\alpha}$ that do not belong to the highest weight representation Λ.
Proof: The MWDF Eq. (5.1.23) implies that

$$\begin{aligned}\sum_{K=-\infty}^{\infty} n_{M+\alpha K}\,\langle M+K\alpha,\alpha^\vee\rangle = &\sum_{K=-\infty}^{\infty} n_{M+\alpha K}\,[(m-p)_{M;\alpha}+2K]\\ =& n_{M+p\alpha}\,[(m-p)+2p] + n_{M-m\alpha}\,[(m-p)-2m]\\ &+\\ &n_{M+(p-1)\alpha}\,[(m-p)+2(p-1)] + n_{M-(m-1)\alpha}\,[(m-p)-2(m-1)]\\ &+\dots\\ =& n_{M+p\alpha}\,[2(m-p)+2p-2m]\\ &+\\ &n_{M+(p-1)\alpha}\,[2(m-p)+2(p-1)-2(m-1)]+\dots\\ =&0.\end{aligned} \tag{9.4.23}$$

Here we have used invariance of the weight space dimensions under the S_α Weyl reflection. Thus Eq. (9.4.21) can be written as

$$\begin{aligned}[\langle\Lambda,\Lambda\rangle - \langle M,M\rangle + \langle 2\rho_{\rm w},\Lambda-M\rangle]\,n_M &= \sum_{\alpha>0}\sum_{K\geq1} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle - \sum_{\alpha>0} n_M\,\langle M,\alpha\rangle - \sum_{\alpha>0}\sum_{K\leq -1} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle\\ &= \sum_{\alpha>0}\sum_{K\geq1} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle + \sum_{\alpha<0}\sum_{K\geq1} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle - 2n_M\,\langle M,\rho_{\rm w}\rangle,\\ &= \sum_{\alpha\neq0}\sum_{K\geq1} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle - 2n_M\,\langle M,\rho_{\rm w}\rangle.\end{aligned} \tag{9.4.24}$$

We can add the $K=0$ term to the right-hand side, since

$$\sum_{\alpha\neq0} n_M\,\langle M,\alpha\rangle = n_M\;[\langle M,2\rho_{\rm w}\rangle - \langle M,2\rho_{\rm w}\rangle] = 0.$$

Therefore

$$[\langle\Lambda+\rho_{\rm w},\Lambda+\rho_{\rm w}\rangle - \langle\rho_{\rm w},\rho_{\rm w}\rangle - \langle M,M\rangle]\,n_M = \sum_{\alpha\neq0}\sum_{K\geq0} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle. \tag{9.4.25}$$

9.4.3 Resumming Freudenthal

Multiplying both sides of Eq. (9.4.25) by $\exp[\langle M,\beta\rangle]$ and summing over M, we get

$$[\langle\Lambda+\rho_{\rm w},\Lambda+\rho_{\rm w}\rangle - \langle\rho_{\rm w},\rho_{\rm w}\rangle]\,\chi(\beta) - \sum_M \langle M,M\rangle n_M e^{\langle M,\beta\rangle} = \sum_M\sum_{\alpha\neq0}\sum_{K\geq0} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle\, e^{\langle M,\beta\rangle}. \tag{9.4.26}$$

If we express the weight β in terms of the overcomplete sum over all roots $\alpha\in\Delta$

$$\beta \equiv \sum_{\alpha\in\Delta}\beta_\alpha\,\alpha, \tag{9.4.27}$$

then

$$\frac{\partial^p}{\partial(\beta_\alpha)^p} e^{\langle M,\beta\rangle} = \langle M,\alpha\rangle^p e^{\langle M,\beta\rangle} \quad \Rightarrow \quad \sum_{\alpha\in\Delta} \frac{\partial^2}{\partial(\beta_\alpha)^2} e^{\langle M,\beta\rangle} = \sum_{\alpha\in\Delta} \langle M,\alpha\rangle\langle\alpha,M\rangle e^{\langle M,\beta\rangle}. \tag{9.4.28}$$

Now, recall that the scalar product between two weights is defined via the Killing form:

$$\langle M,N\rangle = (h_M, h_N).$$

The Killing form can be evaluated by summing over all elements of the Lie algebra [Eq. (4.2.2)],

$$(x,y) = \sum_{\overline{\alpha}_i} [x,[y,h_{\overline{\alpha}_i}]]|_{h_{\overline{\alpha}_i}} + \sum_{\alpha\neq 0} [x,[y,e_\alpha]]|_{e_\alpha}, \tag{9.4.29}$$

where $\mathcal{O}|_x$ means take the coefficient of the Lie algebra x. Thus

$$\langle M,N\rangle = (h_M,h_N) = \sum_{\alpha\neq 0} \alpha(h_M)\,\alpha(h_N) = \sum_{\alpha\neq 0} \langle M,\alpha\rangle\langle\alpha,N\rangle. \tag{9.4.30}$$

Therefore Eq. (9.4.28) implies that

$$\sum_{\alpha\in\Delta} \frac{\partial^2}{\partial(\beta_\alpha)^2} e^{\langle M,\beta\rangle} = \langle M,M\rangle\, e^{\langle M,\beta\rangle}, \tag{9.4.31}$$

and Eq. (9.4.26) becomes

$$\left[\langle\Lambda+\rho_{\mathrm{W}},\Lambda+\rho_{\mathrm{W}}\rangle - \langle\rho_{\mathrm{W}},\rho_{\mathrm{W}}\rangle - \sum_{\alpha\in\Delta}\frac{\partial^2}{\partial(\beta_\alpha)^2}\right]\chi(\beta) = \sum_{\alpha\neq 0}\sum_{M} I_{M,\alpha}(\beta), \tag{9.4.32a}$$

$$I_{M,\alpha}(\beta) \equiv \sum_{K\geq 0} n_{M+\alpha K}\,\langle M+K\alpha,\alpha\rangle\, e^{\langle M,\beta\rangle}. \tag{9.4.32b}$$

The left-hand side of Eq. (9.4.32a) is a differential equation for the character function $\chi(\beta)$, viewed as a function of the "coordinates" $\{\beta_\alpha\}$ in Eq. (9.4.27). We want to express the right-hand side of Eq. (9.2.10) in terms of the generating functions $\chi(\beta)$ and $Q(\beta)$, Eq. (9.4.7).

Consider $I_{M,\alpha}(\beta)$, Eq. (9.4.32b). For a fixed weight M in the weight string generated by $\hat{E}_{\pm\alpha}$, we must determine the contribution of each spin-j representation of this su(2) [Eq. (9.2.2)] to $I_{M,\alpha}(\beta)$. The spin-j representation has an odd ($j\in\mathbb{Z}$) or even [$j\in(2\mathbb{Z}+1)/2$] number of weights. Let M_0 locate the center of this representation. For integer j ($2j+1$ odd), M_0 is the center weight; for half-integer j ($2j+1$ even), M_0 is not itself a weight. Then a generic weight M lying somewhere in this representation can be written as

$$M = M_0 + q\alpha, \qquad q\in\{-j,-j+1,\ldots,j-1,j\}. \tag{9.4.33}$$

Suppose one copy of the spin-j representation with some (undetermined) M_0 contributes to $I_{M,\alpha}(\beta)$ in Eq. (9.4.32b). The contribution is

$$I^{(j)}_{M,\alpha}(\beta) \equiv \sum_{K\geq 0}^{j-q} (1)\,\langle M_0+(K+q)\alpha,\alpha\rangle\, e^{\langle M_0+q\alpha,\beta\rangle} = \langle\alpha,\alpha\rangle\, e^{\langle M_0,\beta\rangle} \sum_{K\geq 0}^{j-q} (K+q)\, e^{q\langle\alpha,\beta\rangle}, \tag{9.4.34}$$

where we've used the fact that $\langle M_0,\alpha\rangle = 0$; the latter is obvious for the case where M_0 is itself the center weight [Eq. (5.1.23)]. We can sum over the nondegenerate weights [Eq. (9.4.33)] in this spin-j representation to get

$$I^{(j)}_{\alpha}(\beta) \equiv \sum_{M} I^{(j)}_{M,\alpha}(\beta) = \langle\alpha,\alpha\rangle\, e^{\langle M_0,\beta\rangle} \sum_{q=-j}^{j}\sum_{K\geq 0}^{j-q} (K+q)\, e^{q\langle\alpha,\beta\rangle}. \tag{9.4.35}$$

Consider

$$\left[e^{\langle\alpha,\beta\rangle}-1\right] I_\alpha^{(j)}(\beta) = \langle\alpha,\alpha\rangle\, e^{\langle M_0,\beta\rangle}\left[\sum_{q=-j}^{j}\sum_{K\geq 0}^{j-q}(K+q)\,e^{(q+1)\langle\alpha,\beta\rangle} - \sum_{q=-j-1}^{j-1}\sum_{K\geq 0}^{j-q-1}(K+q+1)\,e^{(q+1)\langle\alpha,\beta\rangle}\right]$$

$$= \langle\alpha,\alpha\rangle\, e^{\langle M_0,\beta\rangle}\left[\begin{array}{l} j\,e^{(j+1)\langle\alpha,\beta\rangle} + \sum_{q=-j}^{j-1}\sum_{K\geq 0}^{j-q-1}(K+q)\,e^{(q+1)\langle\alpha,\beta\rangle} + \sum_{q=-j}^{j-1} j\,e^{(q+1)\langle\alpha,\beta\rangle} \\ -\sum_{q=-j}^{j-1}\sum_{K\geq 0}^{j-q-1}(K+q+1)\,e^{(q+1)\langle\alpha,\beta\rangle} - \sum_{K\geq 0}^{2j}(K-j)\,e^{-j\langle\alpha,\beta\rangle}\end{array}\right] \tag{9.4.36}$$

$$= \langle\alpha,\alpha\rangle\, e^{\langle M_0,\beta\rangle}\left[j\,e^{(j+1)\langle\alpha,\beta\rangle} + \sum_{q=-j}^{j-1} j\,e^{(q+1)\langle\alpha,\beta\rangle} - \sum_{q=-j}^{j-1}(j-q)\,e^{(q+1)\langle\alpha,\beta\rangle}\right]$$

$$= \langle\alpha,\alpha\rangle\, e^{\langle M_0,\beta\rangle}\sum_{q=-j}^{j} q\,e^{(q+1)\langle\alpha,\beta\rangle}.$$

Thus

$$I_\alpha^{(j)}(\beta) = \frac{1}{\left[e^{\langle\alpha,\beta\rangle}-1\right]}\sum_{q=-j}^{j}\langle q\alpha,\alpha\rangle\, e^{\langle M_0+q\alpha+\alpha,\beta\rangle} = \frac{1}{\left[e^{\langle\alpha,\beta\rangle}-1\right]}\sum_{q=-j}^{j}\langle M,\alpha\rangle\, e^{\langle M+\alpha,\beta\rangle}, \tag{9.4.37}$$

where we have used Eq. (9.4.33) and the fact that $\langle M_0,\alpha\rangle = 0$. Eq. (9.4.37) is the contribution to the right-hand side of Eq. (9.4.32a) of *all weights* in the spin-j su(2) generated by $\hat{E}_{\pm\alpha}$, with central weight M_0. **The key result is that Eq. (9.4.37) does not depend explicitly on M_0, but only implicitly through the (summed) weight M [Eq. (9.4.33)].**

We can therefore rewrite the right-hand side of Eq. (9.4.32a) as

$$\text{RHS} \equiv \sum_{\alpha\neq 0}\sum_{M} I_{M,\alpha}(\beta) = \sum_{\alpha\neq 0}\sum_{M_0}\sum_{j} n_{M_0,j}^{(\alpha)}\, I_\alpha^{(j)}(\beta). \tag{9.4.38}$$

For each root α, we must sum over all central weights M_0 and all representations j; there is an (undetermined) degeneracy factor $n_{M_0,j}^{(\alpha)}$ that can be zero, one, or more if there are multiple spin-j su(2) representations with the same central weight that contribute. At this point it looks like we have just introduced many more unknowns, but now a minor miracle ensues. Combining Eqs. (9.4.37) and (9.4.38), we get

$$\text{RHS} = \sum_{\alpha\neq 0}\frac{1}{\left[e^{\langle\alpha,\beta\rangle}-1\right]}\sum_{M_0}\sum_{j}\sum_{q=-j}^{j} n_{M_0,j}^{(\alpha)}\,\langle M,\alpha\rangle\, e^{\langle M+\alpha,\beta\rangle} = \sum_{\alpha\neq 0}\frac{1}{\left[e^{\langle\alpha,\beta\rangle}-1\right]}\sum_{M} n_M\,\langle M,\alpha\rangle\, e^{\langle M+\alpha,\beta\rangle}, \tag{9.4.39}$$

since the sum over representation centers M_0, representations j, and weights $q\in\{-j,\ldots,j\}$ weighted by the degeneracy $n_{M_0,j}^{(\alpha)}$ is just a particular decomposition of the sum over all weights M (again weighted by the appropriate degeneracy).

As an example, consider the (2,1) representation of su(3) shown in Fig. 9.2. For simple root $\overline{\alpha}_1$, the center weights $\{M_0\}$ enumerate the four parallel horizontal weight strings. Each horizontal string consists of one or two different spin-j representations. Summing over all M_0, all j, and states within each j is equivalent to summing over all weights times the appropriate degeneracies. Note that in this case, $n_{M_0,j}^{(\overline{\alpha}_1)} = 1$ for all M_0 and j.

9.4.4 *finis*

Replacing the right-hand side of Eq. (9.4.32a) with Eq. (9.4.39), we have

$$\left[\langle\Lambda+\rho_{\mathsf{W}},\Lambda+\rho_{\mathsf{W}}\rangle - \langle\rho_{\mathsf{W}},\rho_{\mathsf{W}}\rangle - \sum_{\alpha\in\Delta}\frac{\partial^2}{\partial(\beta_\alpha)^2}\right]\chi(\beta) = \sum_{\alpha\neq 0}\sum_{M}\frac{n_M\,\langle M,\alpha\rangle\, e^{\langle M+\alpha,\beta\rangle}}{\left[e^{\langle\alpha,\beta\rangle}-1\right]}. \tag{9.4.40}$$

Eq. (9.4.7b) implies that

$$Q^2(\beta) = \prod_{\alpha>0} \left[e^{\langle\alpha,\beta\rangle} - 1\right] \left[1 - e^{-\langle\alpha,\beta\rangle}\right] = \prod_{\alpha\neq 0} \left[e^{\langle\alpha,\beta\rangle} - 1\right] (-1)^{|\Delta_+|}, \tag{9.4.41}$$

where $|\Delta_+|$ denotes the number of positive roots. Then

$$\frac{\partial}{\partial\beta_\alpha} \log[Q^2(\beta)] = \sum_{\alpha'\neq 0} \frac{\langle\alpha',\alpha\rangle\, e^{\langle\alpha',\beta\rangle}}{\left[e^{\langle\alpha',\beta\rangle} - 1\right]}, \tag{9.4.42}$$

where β_α is an expansion coefficient for weight β in the overcomplete set of all roots, Eq. (9.4.27). Combining the above with Eq. (9.4.28), we get

$$\begin{aligned}\sum_{\alpha\neq 0} \left\{\frac{\partial}{\partial\beta_\alpha} \log[Q^2(\beta)]\right\} \left\{\frac{\partial}{\partial\beta_\alpha}\chi(\beta)\right\} &= \sum_M n_M \sum_{\alpha'\neq 0}\sum_{\alpha\neq 0} \frac{\langle\alpha',\alpha\rangle\langle\alpha, M\rangle e^{\langle M+\alpha',\beta\rangle}}{\left[e^{\langle\alpha',\beta\rangle} - 1\right]} \\ &= \sum_M n_M \sum_{\alpha'\neq 0} \frac{\langle\alpha', M\rangle e^{\langle M+\alpha',\beta\rangle}}{\left[e^{\langle\alpha',\beta\rangle} - 1\right]} \\ &= \frac{1}{Q}\sum_{\alpha\neq 0}\left[\frac{\partial^2}{\partial(\beta_\alpha)^2}\left(Q\chi\right)(\beta) - \chi\frac{\partial^2}{\partial(\beta_\alpha)^2}Q(\beta) - Q\frac{\partial^2}{\partial(\beta_\alpha)^2}\chi(\beta)\right].\end{aligned} \tag{9.4.43}$$

On the second line, we have used Eq. (9.4.30). Eq. (9.4.40) becomes

$$\left[\langle\Lambda+\rho_{\mathsf{W}}, \Lambda+\rho_{\mathsf{W}}\rangle - \langle\rho_{\mathsf{W}}, \rho_{\mathsf{W}}\rangle\right]\left(Q\chi\right)(\beta) = \sum_{\alpha\neq 0}\left[\frac{\partial^2}{\partial(\beta_\alpha)^2}\left(Q\chi\right)(\beta) - \chi\frac{\partial^2}{\partial(\beta_\alpha)^2}Q(\beta)\right]. \tag{9.4.44}$$

Eq. (9.4.20) implies that

$$\begin{aligned}\sum_{\alpha\neq 0} \frac{\partial^2}{\partial(\beta_\alpha)^2}Q(\beta) &= \sum_{\alpha\neq 0}\sum_{w\in W} (\det w)\,\langle w\,\rho_{\mathsf{W}}, \alpha\rangle\langle\alpha, w\,\rho_{\mathsf{W}}\rangle\, e^{\langle w\,\rho_{\mathsf{W}},\beta\rangle} \\ &= \sum_{w\in W} (\det w)\,\langle w\,\rho_{\mathsf{W}}, w\,\rho_{\mathsf{W}}\rangle\, e^{\langle w\,\rho_{\mathsf{W}},\beta\rangle} \\ &= \langle\rho_{\mathsf{W}}, \rho_{\mathsf{W}}\rangle\, Q(\beta).\end{aligned} \tag{9.4.45}$$

Therefore Eq. (9.4.44) becomes

$$\sum_{\alpha\neq 0} \frac{\partial^2}{\partial(\beta_\alpha)^2}\left(Q\chi\right)(\beta) = \langle\Lambda+\rho_{\mathsf{W}}, \Lambda+\rho_{\mathsf{W}}\rangle\ \left(Q\chi\right)(\beta). \tag{9.4.46}$$

Note that $(Q\chi)(\beta)$ is an alternating function [Eqs. (9.4.8) and (9.4.11)]. Eqs. (9.4.46), (9.4.45), and (9.4.20) suggest the following simple ansatz (guess):

$$\left(Q\chi\right)(\beta) = \sum_{w\in W} (\det w)\, e^{\langle\Lambda+\rho_{\mathsf{W}}, w\,\beta\rangle}. \tag{9.4.47}$$

This obviously satisfies Eq. (9.4.46), but it is not the unique solution. The precise form of Eq. (9.4.47) can be proven following a similar line of argument used to obtain Eq. (9.4.20) from Eqs. (9.4.15) and (9.4.16).

Combining Eqs. (9.4.47) and (9.4.20), we finally obtain

$$\chi(\beta) = \sum_M n_M e^{\langle M,\beta\rangle} = \frac{\sum_{w\in W} (\det w)\, e^{\langle\Lambda+\rho_{\mathsf{W}}, w\,\beta\rangle}}{\sum_{w\in W} (\det w)\, e^{\langle\rho_{\mathsf{W}}, w\,\beta\rangle}}, \qquad \text{Weyl's character formula.} \tag{9.4.48}$$

Eq. (9.4.48) is essentially a resummation of Freudenthal's recursion formula Eq. (9.2.10), assisted by the Weyl group.

9.5 Dimension and strange formulae

The Weyl character χ is a functional defined for any irreducible representation; in Eq. (9.4.48) it is evaluated for an arbitrary element $\beta \in H_0^*$. If we set

$$\beta = t\,\rho_{\mathrm{W}} \tag{9.5.1}$$

with $t \in \mathbb{R}$, then using Eq. (9.4.7b)

$$\begin{aligned}\chi\left(t\,\rho_{\mathrm{W}}\right) &= \frac{\sum_{w\in W} (\det w)\, e^{\langle w\,\rho_{\mathrm{W}}, t(\Lambda+\rho_{\mathrm{W}})\rangle}}{\sum_{w\in W} (\det w)\, e^{\langle w\,\rho_{\mathrm{W}}, t\,\rho_{\mathrm{W}}\rangle}} = \frac{Q\left[t\left(\Lambda+\rho_{\mathrm{W}}\right)\right]}{Q\left(t\,\rho_{\mathrm{W}}\right)} \\ &= \prod_{\alpha>0} \frac{\sinh\left[\frac{1}{2}\langle\alpha, t\left(\Lambda+\rho_{\mathrm{W}}\right)\rangle\right]}{\sinh\left[\frac{1}{2}\langle\alpha, t\,\rho_{\mathrm{W}}\rangle\right]}.\end{aligned} \tag{9.5.2}$$

Therefore

$$\lim_{t\to 0} \chi\left(t\,\rho_{\mathrm{W}}\right) = \sum_M n_M = N_\Lambda \simeq \lim_{t\to 0} \frac{\prod_{\alpha>0} \langle\alpha, t\left(\Lambda+\rho_{\mathrm{W}}\right)\rangle}{\prod_{\alpha>0} \langle\alpha, t\,\rho_{\mathrm{W}}\rangle}, \tag{9.5.3}$$

where N_Λ is the dimension of the representation. We thus have the **dimension formula**

$$N_\Lambda = \prod_{\alpha>0} \frac{\langle\alpha, \Lambda+\rho_{\mathrm{W}}\rangle}{\langle\alpha, \rho_{\mathrm{W}}\rangle}, \qquad \text{Dimension } N_\Lambda \text{ of highest weight representation } \Lambda. \tag{9.5.4}$$

We can obtain another formula from Eq. (9.5.2). Expanding again for small t,

$$\begin{aligned}\chi\left(t\,\rho_{\mathrm{W}}\right) &\simeq \prod_{\alpha>0} \frac{\langle\alpha, \left(\Lambda+\rho_{\mathrm{W}}\right)\rangle\left[1+\frac{t^2}{24}\langle\alpha, \left(\Lambda+\rho_{\mathrm{W}}\right)\rangle^2+\ldots\right]}{\langle\alpha, \rho_{\mathrm{W}}\rangle\left[1+\frac{t^2}{24}\langle\alpha, \rho_{\mathrm{W}}\rangle^2+\ldots\right]} \\ &= N_\Lambda\left\{1+\frac{t^2}{48}\sum_{\alpha\neq 0}\left[\langle\left(\Lambda+\rho_{\mathrm{W}}\right), \alpha\rangle\langle\alpha, \left(\Lambda+\rho_{\mathrm{W}}\right)\rangle - \langle\rho_{\mathrm{W}}, \alpha\rangle\langle\alpha, \rho_{\mathrm{W}}\rangle\right]+\ldots\right\} \\ &= N_\Lambda\left\{1+\frac{t^2}{48}\left[\langle\left(\Lambda+\rho_{\mathrm{W}}\right), \left(\Lambda+\rho_{\mathrm{W}}\right)\rangle - \langle\rho_{\mathrm{W}}, \rho_{\mathrm{W}}\rangle\right]+\ldots\right\},\end{aligned} \tag{9.5.5}$$

where we have used Eq. (9.4.30). On the other hand,

$$\chi(t\,\rho_{\mathrm{W}}) = \sum_M n_M e^{\langle M, t\,\rho_{\mathrm{W}}\rangle} = \sum_M n_M\left[1+t\langle M,\ \rho_{\mathrm{W}}\rangle+\frac{t^2}{2!}\langle M,\ \rho_{\mathrm{W}}\rangle^2+\ldots\right] \tag{9.5.6}$$

The terms linear in t sum to zero due to Weyl invariance. If we take $\Lambda = \theta$ (the highest root, i.e. we are studying the adjoint representation), then the sum over M is replaced by a sum over non-zero roots (since the inner product with the zero weight specifying the Cartan subalgebra gives zero). Therefore

$$\chi(t\,\rho_{\mathrm{W}}) = N_\theta + \frac{t^2}{2!}\sum_{\alpha\neq 0}\langle\alpha,\ \rho_{\mathrm{W}}\rangle^2 + \ldots = N_\theta + \frac{t^2}{2}\langle\rho_{\mathrm{W}}, \rho_{\mathrm{W}}\rangle + \ldots, \tag{9.5.7}$$

where we have used Eq. (9.4.30) and the non-degeneracy of the roots $n_\alpha = 1$ (Proposition (VI.), Sec. 5.2.2]. Eqs. (9.5.5) and (9.5.7) imply that

$$\langle\rho_{\mathrm{W}}, \rho_{\mathrm{W}}\rangle = \frac{N_\theta}{24}\langle\theta, \theta+2\rho_{\mathrm{W}}\rangle. \tag{9.5.8}$$

Here $N_\theta = d$ is the dimension of the adjoint representation (or the Lie group). In the standard normalization we obtain

$$\langle \rho_{\mathsf{W}}, \rho_{\mathsf{W}} \rangle_2 = \frac{N_\theta\, g}{12}, \qquad \text{Freudenthal-de Vries strange formula.} \tag{9.5.9}$$

Here g denotes the dual Coxeter number [Eq. (9.1.21)].

9.5.1 Dimensions for generic representations of the classical algebras

There are different methods to evaluate Eq. (9.5.4) for a generic highest weight representation Λ in a particular Lie algebra. We will use a strategy that works for all algebras and can be easily evaluated numerically.

The simplest case (unsurprisingly) is $A_{n-1} = \mathrm{su}(n)$. In Sec. 7.1, we showed that the positive roots can be enumerated $\{\alpha_{ij}\}$ with $1 \le i < j \le n$. These are expressed in terms of the simple roots via Eq. (7.1.11),

$$\alpha_{ij} = \sum_{k=i}^{j-1} \overline{\alpha}_k = \sum_{k=i}^{j-1} \overline{\alpha}_k^\vee. \tag{9.5.10}$$

Then

$$\langle \Lambda + \rho_{\mathsf{W}}, \alpha_{ij} \rangle = \sum_{k=i}^{j-1} (\Lambda_k + 1) = (j - i) + \sum_{k=i}^{j-1} \Lambda_k \equiv 2\,\mathcal{F}(\Lambda; i, j). \tag{9.5.11a}$$

Therefore

$$N_\Lambda = \prod_{i=1}^{n-1} \prod_{j>i}^{n} \left[\frac{\mathcal{F}(\Lambda; i, j)}{\mathcal{F}(0; i, j)} \right], \qquad \text{Dimension formula for } \mathrm{su}(n) = A_{n-1}. \tag{9.5.12}$$

• **Exercise**: Use Eq. (9.5.12) to show that the $\Lambda = (p, q)$ representation of su(3) has $N_{p,q}$ given by Eq. (9.1.43).

• **Exercise**: Use Eq. (9.5.12) to show that the $\Lambda = \omega_p$ representation of su(n) has dimension

$$N_p = \frac{n!}{p!(n-p)!}, \tag{9.5.13}$$

consistent with a rank-p antisymmetric tensor [c.f. Eq. (8.2.7)].

The dimension formula for sp($2n$) can be similarly obtained using Eqs. (7.1.11) and (7.2.12). The result is

$$N_\Lambda = \prod_{i=1}^{n} \left[\frac{\mathcal{F}(\Lambda; i, n+1)}{\mathcal{F}(0; i, n+1)} \right] \prod_{j>i}^{n} \left\{ \frac{\mathcal{F}(\Lambda; i, j)\,[\mathcal{F}(\Lambda; i, j) + 2\mathcal{F}(\Lambda; j, n+1)]}{\mathcal{F}(0; i, j)\,[\mathcal{F}(0; i, j) + 2\mathcal{F}(0; j, n+1)]} \right\}, \qquad \text{Dimension formula for } \mathrm{sp}(2n) = C_n. \tag{9.5.14}$$

• **Exercise**: Derive Eq. (9.5.14).

• **Exercise**: Use Eq. (9.5.14) to verify that the fundamental representation $\Lambda = \omega_p$ ($1 \le p \le n$) has dimension

$$N_{\omega_p} = 2(n + 1 - p)\frac{(2n + 1)!}{p!(2n - p + 2)!}, \tag{9.5.15}$$

consistent with a "traceless" fully antisymmetric rank-p tensor [Eq. (8.3.11)].

We can also obtain generic dimension formulae for $\mathrm{so}(2n) = D_n$ and $\mathrm{so}(2n+1) = B_n$, but we won't transcribe the results here. The most important are the dimensions of the fundamental spinor representations, which are $N_{\omega_n} = 2^n$ for the ω_n representation of B_n, and $N_{\omega_{n-1}} = N_{\omega_n} = 2^{n-1}$ for the independent ω_{n-1} and ω_n representations of D_n.

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