Termination of typical wavefunction multifractal spectra at the Anderson metal-insulator transition

Matthew S. Foster,¹,² Shinsei Ryu,³ Andreas W. W. Ludwig⁴

¹ Rutgers, the State University of New Jersey
² Columbia University
³ University of California, Berkeley
⁴ University of California, Santa Barbara

June 12th, 2009

Reference arXiv: 0901.0284
Multifractal: An object characterized by a continuous spectrum of fractal dimensions

- Set $S$ of dimension $d$, broken up into $N$ pieces ("boxes") $\Omega_i$

  Box size $a \ll$ system size $L$; $N = (L/a)^d$

- Some probability measure $d\mu$

- Box probability: $\mu_i \equiv \int_{\Omega_i} d\mu$; $\sum_{i=1}^{N} \mu_i = 1$
Multifractal: An object characterized by a continuous spectrum of fractal dimensions

- Set $S$ of dimension $d$, broken up into $N$ pieces ("boxes") $\Omega_i$

  Box size $a \ll$ system size $L$; $N = (L/a)^d$

- Some probability measure $d\mu$

- Box probability: $\mu_i \equiv \int_{\Omega_i} d\mu$; $\sum_{i=1}^{N} \mu_i = 1$

Anderson localization physics (non-interacting electrons):

$$d\mu = |\psi(\mathbf{r})|^2 d^d\mathbf{r}; \quad \left[ -\frac{\nabla^2}{2m} + v(\mathbf{r}) \right] \psi(\mathbf{r}) = \varepsilon \psi(\mathbf{r})$$

Inverse Participation Ratio (IPR):

$$P_q = \sum_{i=1}^{N} \mu_i^q \sim \left( \frac{a}{L} \right)^{\tau(q)}$$
Inverse Participation Ratio (IPR):

\[ P_q \sim \int_{L^d} |\psi(r)|^{2q} \, d^d r \sim \left( \frac{a}{L} \right)^{\tau(q)} \]

- Compute for typical \( \psi \) in a system of size \( L^d \), obtained in a fixed realization of the disorder potential (i.e. numerics)

- Analytical properties:

\[ \frac{d\tau(q)}{dq} \geq 0; \quad \frac{d^2\tau(q)}{dq^2} \leq 0; \quad \lim_{q \to \pm \infty} \frac{\tau(q)}{q - 1} = d_{\pm \infty} \]

- As a probe of wavefunction (de)localization:

  (a) Extended plane wave state

  \[ \tau(q) = d(q - 1) \]
Inverse Participation Ratio (IPR):

\[ P_q \sim \int_{L^d} |\psi(r)|^{2q} \, d^d r \sim \left( \frac{a}{L} \right)^{\tau(q)} \]

- **Compute for typical** \( \psi \) **in a system of size** \( L^d \), **obtained in a fixed realization of the disorder potential** (i.e. numerics)

- **Analytical properties:**

\[ \frac{d\tau(q)}{dq} \geq 0; \quad \frac{d^2\tau(q)}{dq^2} \leq 0; \quad \lim_{q \to \pm\infty} \frac{\tau(q)}{q - 1} = d_{\pm\infty} \]

- **As a probe of wavefunction (de)localization:**

  (b) Exponentially localized state

\[ \tau(q) \sim 0, \quad L \gg \xi_{\text{loc}} \]
Critical wavefunctions at a delocalization transition (Anderson MIT in $d > 2$, IQHP in $d = 2$)

- $\psi$ exhibits **multifractal scaling**
  (neither localized nor extended, nor a simple fractal)
  Wegner 1980; Castellani and Peliti 1986

- **Multifractality encoded in nonlinearity**
  of the $\tau(q)$ spectrum

- $\tau(q)$ is **self-averaging and universal**
  e.g., Pook, Janssen 1991; Huckestein 1995; Evers, Mildenberger, Mirlin 2001
  Chamon, Mudry, Wen 1996; Mirlin and Evers 2000
  Vasquez, Rodriguez, Roemer 2008
“All states are localized in $d = 2$”

**Mobility edge/MIT in $d > 2$**

**MIT is continuous (2nd order) QPT, critical conductance $G = G^*$, independent of $L$**

**Average local density of states (LDOS) is non-critical**

**Conventional analytical framework:** $\epsilon$-expansion about $d = 2 + \epsilon$, using the non-linear sigma model (NL$\sigma$M) effective field theory

Abrahams, Anderson, Licciardello, Ramakrishnan 1979

Wegner 1981, McKane and Stone 1981

Wegner 1979
Standard result for the unitary class (disordered metal with broken TRI):

\[ \tilde{\tau}(q) = d(q - 1) - \sqrt{\frac{\epsilon}{2}}q(q - 1) + O\left[\epsilon^2 q^2(q - 1)^2\right] \]

Wegner 1980, 1987; Pruisken 1985

Something is rotten about this result:

- \( \tilde{\tau}(q) \) is not monotonically increasing
- \( \tilde{\tau}(q) \) becomes NEGATIVE for large, positive \( q \gg 1 \)
Average vs. Typical Multifractal Spectra

Standard result for the unitary class (disordered metal with broken TRI):

\[ \tilde{\tau}(q) = d(q - 1) - \frac{\epsilon}{2} q(q - 1) + O \left[ \epsilon^2 q^2 (q - 1)^2 \right] \]

Wegner 1980, 1987; Pruisken 1985

We must take a closer look at the NLσM method:

Typical spectrum defined for representative wavefunction in a fixed disorder realization

Analytical calculation: Average over disorder realizations!

\[ \tilde{\tau}(q) = - \frac{d \ln \overline{P}_q}{d \ln L} \]

Average spectrum

\[ \tau(q) = - \frac{d \ln P_q}{d \ln L} \]

Typical spectrum
Extracting the typical multifractal spectrum from the LDOS

IPR in terms of the local density of states (LDOS):

\[ P_q = \int |\psi(r)|^{2q} d^d r \sim \frac{\int \nu^q(r) d^d r}{\left[ \int \nu(r) d^d r \right]^q} \]

where

\[ \nu(r) = \sum_{\alpha} |\psi_\alpha(r)|^2 \delta(\varepsilon - \varepsilon_\alpha) \]

Cumulant function for the \( q^{th} \) moment of the integrated LDOS:

\[ F_q(\xi; L) \equiv \left\langle e^{-\xi \int d^d r \nu^q(r)} \right\rangle \]
Extracting the typical multifractal spectrum from the LDOS

• **Typical spectrum:**

\[
\tau(q) = \frac{d}{d \ln L} \int_0^\infty \frac{d\xi}{\xi} [F_q(\xi; L) - qF_1(\xi; L)]
\]

• **Average spectrum:**

\[
\tilde{\tau}(q) = d(q - 1) + x_q - q x_1
\]

where \( x_q \) is the **scaling dimension** of \( \nu^q(r) \) at the MIT in \( d = 2 + \epsilon \):

\[
\langle \nu^q(r) \rangle \sim \frac{1}{L^{x_q}}
\]

Cumulant function for the \( q^{th} \) moment of the **integrated** LDOS:

\[
F_q(\xi; L) \equiv \left\langle e^{-\xi \int d^d r \, \nu^q(r)} \right\rangle
\]
Effective low energy field theory: the NL_{oM}

\[ Z = \int \mathcal{D}[\hat{Q}] e^{-S} \]

\[ S[\hat{Q}] = \frac{1}{2t} \int d^d r \ \text{Tr} \ (\nabla \hat{Q} \cdot \nabla \hat{Q}) - h \int d^d r \ \nu(r) \]

where \( \hat{Q}(r) \) is an \( 2n \times 2n \) matrix field satisfying

\[ \hat{Q}^\dagger = \hat{Q}, \quad \hat{Q}^2 = \hat{1}, \quad \text{Tr}(\hat{Q}) = 0 \]

**Fermionic replicas:**  \( \hat{Q} \in \frac{G(2n)}{G(n) \times G(n)}, \quad G \in U, \ Sp, \ O \)
Effective low energy field theory: the NLoM

\[ Z = \int \mathcal{D}[\hat{Q}] e^{-S} \]

\[ S[\hat{Q}] = \frac{1}{2t} \int d^d r \, \text{Tr} \left( \nabla \hat{Q} \cdot \nabla \hat{Q} \right) - h \int d^d r \, \nu(r) \]

where \( \hat{Q}(r) \) is an \( 2n \times 2n \) matrix field satisfying

\[ \hat{Q}^\dagger = \hat{Q}, \quad \hat{Q}^2 = \hat{1}, \quad \text{Tr}(\hat{Q}) = 0 \]

Fermionic replicas: \( \hat{Q} \in \frac{G(2n)}{G(n) \times G(n)}, \quad G \in U, \text{Sp}, \text{O} \)

Parameterization: “\( \pi - \sigma \)” coordinates:

\[ \hat{Q} = \begin{bmatrix} (\hat{1} - \hat{W}\hat{W}^\dagger)^{1/2} & \hat{W} \\ \hat{W}^\dagger & -(\hat{1} - \hat{W}^\dagger\hat{W})^{1/2} \end{bmatrix}, \quad \hat{W}(r) \text{ is } n \times n \text{ complex (unconstrained)} \]

Wegner 1979; Efetov, Larkin, Khmelnitskii 1980
The local density of states is represented by the composite operator

\[ \nu(r) \sim \text{Tr} \left( \begin{bmatrix} 1 & 0 \\ 0 & -\hat{1} \end{bmatrix} \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{21} & \hat{Q}_{22} \end{bmatrix} \right) = \text{Tr}(\hat{Q}_{11} - \hat{Q}_{22}) \]

Consider the case of a fixed number of replicas \( n \in \{1, 2, 3, \ldots\} \),

- **2nd order critical point in** \( d = 2 + \epsilon \) **at** \( t = t^*_n, h = 0 \)

- Symmetry between “\( \pi \)” \( (\hat{W}, \hat{W}^\dagger) \) and “\( \sigma \)” \( (\sqrt{1 - \hat{W}\hat{W}^\dagger}, \sqrt{1 - \hat{W}^\dagger\hat{W}}) \)

components restored at the transition
The local density of states is represented by the composite operator

$$\nu(r) \sim \text{Tr} \left( \begin{bmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{bmatrix} \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{21} & \hat{Q}_{22} \end{bmatrix} \right) = \text{Tr}(\hat{Q}_{11} - \hat{Q}_{22})$$

Consider the case of a fixed number of replicas \( n \in \{1, 2, 3, \ldots \} \),

- **2nd order critical point in** \( d = 2 + \epsilon \) at \( t = t_n^*, h = 0 \)

- symmetry between \( \pi \) \((\hat{W}, \hat{W}^\dagger)\) and \( \sigma \) \((\sqrt{\hat{1} - \hat{W}\hat{W}^\dagger}, \sqrt{\hat{1} - \hat{W}^\dagger\hat{W}})\)
  components restored at the transition

At the MIT, we are free to rotate

$$\nu(r) \rightarrow \text{Tr} \left( \begin{bmatrix} 0 & \hat{1} \\ \hat{1} & 0 \end{bmatrix} \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{21} & \hat{Q}_{22} \end{bmatrix} \right) = \text{Tr}(\hat{W} + \hat{W}^\dagger)$$
LDOS Moments

\[ \nu^q(r) \sim \left[ \text{Tr}(\hat{W} + \hat{W}^\dagger) \right]^q \]

**Problem:** Not an eigenoperator; want most relevant component (for a given, fixed q, at the MIT critical point in \( d = 2 + \epsilon \)).

Consider the case \( n = 1 \):

\[ \frac{U(2)}{U(1) \times U(1)} \sim \frac{O(3)}{O(2)} \]

\[ \hat{W} = \pi, \hat{W}^\dagger = \pi^*, \sigma = (1 - \pi^* \pi)^{1/2} \]

**Eigenoperators:** \( Y_{l,m}(\pi, \pi^*, \sigma), \quad x_l = \frac{\epsilon}{2} l(l + 1) \quad \text{at the critical pt in } d = 2 + \epsilon \)

Brezin, Zinn-Justin, Le Guillou 1976
LDOS and its moments in the NLσM

\( n = 1 \), continued...

**Eigenoperators:** \( Y_{l,m}(\pi, \pi^*, \sigma) \), \( x_l = \frac{\epsilon}{2} l(l + 1) \) \( \text{at the critical pt in} \)
\( d = 2 + \epsilon \)

\[ \nu^q \sim (\pi + \pi^*)^q = \sum_{l=0}^{q} \mathcal{O}^{(q)}_l \]

**Addition of angular momentum:**
\[ \mathcal{O}^{(q)}_l = \sum_{m=-l}^{l} \kappa^{(q)}_{l,m} Y_{l,m}(\pi, \pi^*, \sigma) \]
LDOS and its moments in the NLσM

(n = 1, continued…)

**Eigenoperators:** $Y_{l,m}(\pi, \pi^*, \sigma), \quad x_l = \frac{\epsilon}{2} l(l + 1)$

\[
\nu^q \sim (\pi + \pi^*)^q = \sum_{l=0}^{q} O_l^{(q)}
\]

**Addition of angular momentum:**

\[
O_l^{(q)} = \sum_{m=-l}^{l} \kappa_{l,m}^{(q)} Y_{l,m}(\pi, \pi^*, \sigma)
\]

In particular,

\[
O_q^{(q)} = (\pi^q + \ldots + \pi^{*q})
\]

\[
\langle O_q^{(q)} \rangle \sim \frac{1}{L x_q}, \quad x_q = \frac{\epsilon}{2} q(q + 1)
\]

Lowest, highest weight states $Y_q, \mp q$

Largest positive (least relevant) scaling dimension contributing to $\langle \nu^q \rangle$
LDOS and its moments in the NL\(\sigma\)M

\((n = 1, \text{continued}...\))

**Eigenoperators:** \(Y_{l,m}(\pi, \pi^*, \sigma)\), \(x_l = \frac{\epsilon}{2} l(l + 1)\) \(\left[\text{at the critical pt in}\right] d = 2 + \epsilon\)

\(\nu^q \sim (\pi + \pi^*)^q = \sum_{l=0}^{q} \mathcal{O}_{l}^{(q)}\)

**Addition of angular momentum:** \(\mathcal{O}_{l}^{(q)} = \sum_{m=-l}^{l} \kappa_{l,m}^{(q)} Y_{l,m}(\pi, \pi^*, \sigma)\)

In particular, \(\mathcal{O}_{q}^{(q)} = (\pi^q + \ldots + \pi^* q)\)

**OPE at ZERO coupling \((t = 0)\):**

\(\mathcal{O}_{q}^{(q)}(\mathbf{r}) \mathcal{O}_{q'}^{(q')} (\mathbf{r'}) = C_{q,q'}^{q+q'} \mathcal{O}_{q+q'}^{(q+q')} [(\mathbf{r} + \mathbf{r'})/2] + \ldots\)

\((Y_{q,-q} + \ldots)(Y_{q',-q'} + \ldots) = C_{q,q'}^{q+q'} (Y_{q+q',-q-q'} + \ldots)\)
Replica NL\(\sigma\)M: (general \(n, n \to 0\))

\[ \nu^q(r) \sim \left[ \text{Tr}(\hat{W} + \hat{W}^\dagger) \right]^q \]

Lowest weight state in most relevant representation?

\[ \left[ \text{Tr} \left( \hat{W} \right) \right]^q \rightarrow \text{No—still not an eigenoperator} \]

Altshuler, Kratov, Lerner 1986, 1989
LDOS and its moments in the NLσM

**Replica NLσM:** (general $n$, $n \to 0$)

$$\nu^q(r) \sim \left[ \text{Tr}(\hat{W} + \hat{W}^\dagger) \right]^q$$

**Lowest weight state in most relevant representation?**

$$\left[ \text{Tr}(\hat{W}) \right]^q \quad \text{No—still not an eigenoperator}$$

**Most relevant component:**

$$O_q(r) \equiv O_{q [\alpha_1 \alpha_2 \ldots \alpha_q]}^{{\alpha_1 \alpha_2 \ldots \alpha_q}}(r), \quad x_q = -\sqrt{\frac{\epsilon}{2}} q(q-1) + O \left[ \epsilon^2 q^2 (q-1)^2 \right]$$

**where**

$$O_{q [\beta_1 \beta_2 \ldots \beta_q]}^{{\alpha_1 \alpha_2 \ldots \alpha_q}}(r) \equiv \left( \frac{1}{q!} \right)^2 \sum_{P} \text{sgn}(P) \left[ W^{\alpha_1}_{\beta_{P(1)}} \cdots W^{\alpha_q}_{\beta_{P(q)}} \right]$$

Altshuler, Kratsov, Lerner 1986, 1989

Wegner 1980, 1987; Pruisken 1985
Replica NLσM: (general n, n → 0)

\[ \nu^q(r) \sim \left[ \text{Tr}(\hat{W} + \hat{W}^\dagger) \right]^q \]

Lowest weight state in most relevant representation?

\[ \left[ \text{Tr} \left( \hat{W} \right) \right]^q \quad \text{No—still not an eigenoperator} \]

Most relevant component:

\[ O_2 = \frac{\left[ \text{Tr} \left( \hat{W} \right) \right]^2 - \text{Tr} \left( \hat{W}^2 \right)}{(2!)^2} \]

\[ O_3 = \frac{\left[ \text{Tr} \left( \hat{W} \right) \right]^3 - 3 \text{Tr} \left( \hat{W}^2 \right) \text{Tr} \left( \hat{W} \right) + 2 \text{Tr} \left( \hat{W}^3 \right)}{(3!)^2} \]

Altshuler, Kratsov, Lerner 1986, 1989
At the critical point describing the MIT, the expectation of $q^{th}$ LDOS moment scales with system size according to

$$\langle O_q(r) \rangle \sim \frac{1}{L^{x_q}}, \quad x_q = -\sqrt{\frac{\epsilon}{2}} q(q - 1) + \ldots$$

Wegner 1980, 1987; Pruisken 1985

Key points: LDOS moment exponents (scaling dimensions)

1) $x_q$ always negative for $q > 1$—LDOS moment operators always relevant perturbations (in the RG sense, at the MIT)

$$x_q < 0, \quad q > 1$$

2) Convex property: higher moments more relevant than lower ones!

$$x_{q+q'} < x_q + x_{q'} < 0$$
...back to the multifractal spectrum

\[ \langle O_q(r) \rangle \sim \frac{1}{L^{x_q}}, \quad x_q = -\sqrt{\frac{\epsilon}{2}}q(q-1) + \ldots \]

• **Typical spectrum:**

\[ \tau(q) = \left. \frac{d}{d \ln L} \int_0^\infty \frac{d\xi}{\xi} \left[ F_q(\xi; L) - qF_1(\xi; L) \right] \right|_{\ln L=0} \]

• **Average spectrum:**

\[ \tilde{\tau}(q) = d(q - 1) + x_q - qx_1 \]

Cumulant function for the \( q^{th} \) moment of the integrated LDOS:

\[ F_q(\xi; L) \equiv \left\langle e^{-\xi \int d^d r \ O_q(r)} \right\rangle \]
The cumulant expansion is Doomed!

Simplest approach: cumulant expansion

\[ F_q(\xi; L) \sim e^{-\xi \int d^d r \langle O_q(r) \rangle} + \frac{\xi^2}{2!} \left[ \int d^d r d^d r' \langle O_q(r) O_q(r') \rangle_c \right] + \ldots \]

• First term: average is typical
  \[ \tilde{\tau}(q) = \tau(q) \]

• Second term: Operator Product Expansion (OPE) means TROUBLE!

\[ O_q(r) O_{q'}(r') \sim \frac{C^{q+q'}_{q,q'}}{|r - r'|^{x_q + x_{q'} - x_{q+q'}}} O_{q+q'} \left( \frac{r + r'}{2} \right) + \ldots \]

\[ x_q < 0, \quad q > 1 \]
\[ x_{q+q'} < x_q + x_{q'} < 0 \]

Higher cumulants give ever LARGER contributions! Cumulant expansion does not converge!
Following Carpentier and Le Doussal (2000) and Mudry, Ryu, Furusaki (2003), we use the FRG to keep track of all moments:

1. Generalize $F_q$, including all higher moments from very beginning

$$F_q(\xi; L) = \left< e^{\sum_{p=1}^{\infty} y_{pq} \int d^d r \mathcal{O}_{pq}(r)} \right>; \quad y_{pq}(0) = -\xi \delta_{p,1}$$
Following Carpentier and Le Doussal (2000) and Mudry, Ryu, Furusaki (2003), we use the FRG to keep track of all moments:

1. Generalize $F_q$, including all higher moments from very beginning

$$F_q(\xi; L) = \left< e^{\sum_{p=1}^{\infty} y_{pq} \int d^d r \, O_{pq}(r)} \right>; \quad y_{pq}(0) = -\xi \delta_{p,1}$$

2. Study scaling of $F_q$: write RG flow equations for coupling constants

$$\left\{ y_{pq} \right\} \text{ at the MIT in } d = 2 + \epsilon \quad (\text{using the OPE})$$

$$\frac{d y_{pq}}{d l} = (d - x_q) y_{pq} + \frac{S_d}{2} \sum_{m=1}^{p-1} C_{m,p-m}^p y_{mq} y_{p-m,q} + O(y^3)$$

where

$$x_q = -\sqrt{\frac{\epsilon}{2}} q(q - 1) + O\left[ \epsilon^2 q^2 (q - 1)^2 \right] \quad \text{Wegner 1980, 1987; Pruisken 1985}$$

$$C_{q,q'}^{q+q'} = \binom{q + q'}{q} \left[ 1 + O(\epsilon) \right] \quad \text{NEW Foster, Ryu, Ludwig 2009}$$
3. Write auxillary generating functional for the coupling constants

\[ G_q(z, l) \equiv \tilde{G}_q(\tilde{z}, \tilde{l}) \equiv 1 + \frac{S_d}{2d} \sum_{p=1}^{\infty} \frac{(e^{-z})^p}{p!} y_{pq}(l) \]

Initial condition:

\[ G_q(z, 0) \sim \exp \left[ -\xi \frac{S_d}{2d} e^{-z} \right] \]

Galilean boost:

\[ \tilde{z} = z + \sqrt{\frac{\epsilon}{2q}} l, \quad \tilde{l} = l \]
3. Write auxiliary generating functional for the coupling constants

\[ G_q(z, l) = \tilde{G}_q(\tilde{z}, \tilde{l}) = 1 + \frac{S_d}{2d} \sum_{p=1}^{\infty} \frac{(e^{-\tilde{z}})^p}{p!} y_{pq}(l) \]

Initial condition:

\[ G_q(z, 0) \sim \exp \left[ -\xi \frac{S_d}{2d} e^{-\tilde{z}} \right] \]

Galilean boost:

\[ \tilde{z} = z + \sqrt{\frac{\epsilon}{2}} q l, \quad \tilde{l} = l \]

4. \( \tilde{G}_q \) satisfies the Kolmogorov-Petrovsky-Piscounov (KPP) equation, which describes non-linear diffusion in 1D.

\[ \left[ \frac{1}{d} \partial_{\tilde{l}} - D_q \partial^2_{\tilde{z}} \right] \tilde{G}_q = \tilde{G}_q(\tilde{G}_q - 1); \quad D_q = \frac{q^2}{d} \sqrt{\frac{\epsilon}{2}} \]
KPP has propagating front solutions, with a universal front velocity $\tilde{c}_q$. Velocity selection: $\tilde{c}_q$ is a non-analytic function of the diffusion constant $D_q$.

\[ \tilde{G}_q(\tilde{z}, \tilde{l}) \big|_{l \to \infty} \sim h(\tilde{z} - \tilde{c}_q \tilde{l}); \quad \tilde{c}_q = \begin{cases} 
 d (1 + D_q), & D_q \leq 1 \\
 2d \sqrt{D_q}, & D_q > 1
\end{cases} \]
Solution: the Functional Renormalization Group (FRG)

KPP has propagating front solutions, with a universal front velocity $\tilde{c}_q$. Velocity selection: $\tilde{c}_q$ is a non-analytic function of the diffusion constant.

$$\tilde{G}_q(\tilde{z}, \tilde{l}) \big|_{l \to \infty} \sim h(\tilde{z} - \tilde{c}_q \tilde{l})$$

$$\tilde{c}_q = \begin{cases} 
  d(1 + D_q), & D_q \leq 1 \\
  \frac{d}{2d} \sqrt{D_q}, & D_q > 1 
\end{cases}$$

$G_q$ tracks scaling of the coupling constants. FRG: the set $\left\{ y_{pq} \right\}$ “fuses” into single, typical coupling $y_q^{\text{typ}} \sim e^{c_q l}$

Associated typical scaling dimension:

$$x_q^{\text{typ}} \equiv d - c_q$$

$$= \begin{cases} 
  - \frac{d}{q_c^2} q(q - 1), & 1 \leq q \leq q_c \\
  d(1 - q) + q d \left(1 - \frac{\text{sgn}(q)}{q_c}\right)^2, & q > q_c 
\end{cases}$$

$$q_c^2 = 2 \sqrt{\frac{2}{\epsilon}}$$
Evaluate $F_q(\xi; L)$ in the cumulant expansion, \textbf{after coarse-graining (running the RG)}

$$F_q(\xi; L) \sim e^{-y_q^{\text{typ}}} \int d^dr \langle O_q^{\text{typ}}(r) \rangle \sim e^{-\xi L^{d-x_q^{\text{typ}}}}$$

$$\Rightarrow \quad \tau(q) = \frac{d}{d \ln L} \int_0^\infty \frac{d\xi}{\xi} \left[ F_q(\xi; L) - qF_1(\xi; L) \right]$$

$$= d(q - 1) + x_q^{\text{typ}} - q x_1^{\text{typ}}$$

**KPP velocity selection = Termination of the typical multifractal spectrum**

$$\tau(q) = \begin{cases} 
\tilde{\tau}(q) = d(q - 1) \left(1 - \frac{q}{q_c^2}\right), & |q| < q_c \\
 d \left(1 - \frac{\text{sgn}(q)}{q_c}\right)^2 q, & |q| > q_c 
\end{cases}$$
Final result: Termination of the multifractal spectrum at the MIT

KPP velocity selection = Termination of the typical multifractal spectrum

\[
\tau(q) = \begin{cases} 
\tilde{\tau}(q) = d(q - 1) \left(1 - \frac{q}{q_c^2}\right), & |q| < q_c \\
d \left(1 - \frac{\text{sgn}(q)}{q_c}\right)^2 q, & |q| > q_c 
\end{cases}
\]

\[q_c^2 \sim 2 \sqrt{\frac{2}{\epsilon}}\]

Interpretation:

IPR for representative wavefunction, \textit{typical} disorder realization, dominated by rare wavefunction extrema for large $|q|$--simple fractal behavior beyond termination.
Validity of termination within the epsilon expansion:

\[ \tau(q) = \begin{cases} 
\tilde{\tau}(q) & |q| < q_c \\
 d \left( 1 - \frac{\text{sgn}(q)}{q_c} \right)^2 q & |q| > q_c 
\end{cases} \]

\[ q_c^2 \sim 2 \sqrt{\frac{2}{\epsilon}} \]

4-loop result for the scaling dimension (Wegner 1987):

\[ x_q = -\sqrt{\frac{\epsilon}{2}} q(q - 1) - \frac{3\zeta(3)}{8} \epsilon^2 q^2 (q - 1)^2 + O \left( \epsilon^{5/2} \right) \]

At the termination threshold:

\[ x_{q=q_c} = -2 + (2\epsilon)^{1/4} + O \left( \epsilon^{1/2} \right) \]
Although uncontrolled, it is interesting to consider the large-$\epsilon$ limit of the FRG predictions: (strong disorder, high dimensionality)

1. For $\epsilon > \epsilon_F \sim 8$, $q_c < 1$. Then the typical LDOS acquires a non-zero scaling dimension

$$x_1^{typ} = d \left( 1 - \frac{1}{q_c} \right)^2 > 0$$

2. Typical multifractal spectrum exhibits freezing:

$$\tau(q) = \begin{cases} 
- d \left( 1 - \frac{q}{q_c} \right)^2, & |q| \leq q_c \\
\frac{2d}{q_c} (q - |q|), & |q| > q_c 
\end{cases}$$

$$q_c = \frac{1}{2d} \left( 1 - \frac{1}{q_c} \right)^2$$

$$\therefore \tau(q) = 0$$

$q \in \{1, 2, 3, \ldots\}$
Summary

• At a random critical point, entire distribution functions often required to characterize observables: Functional RG method is required

• Typical multifractal spectrum should be universal and self-averaging
  - Extract from FRG for the LDOS moment generating function
  - Key input: the OPE between different moment operators
  - Mapping of FRG flow equations to KPP
  - KPP velocity selection determines termination of the typical spectrum

• Interesting (speculative) connections to high-$d$ Bethe lattice results (see paper for details)
How to extract the Typical multifractal spectrum

Typical spectrum obtains via the LDOS Moment Generating Function $F_q$:

$$\tau(q) = \frac{d}{d \ln L} \int_0^\infty \frac{d\xi}{\xi} \left[ F_q(\xi; L) - qF_1(\xi; L) \right]$$

$$F_q(\xi; L) = \left\langle e^{-\xi \int d^d r \mathcal{O}_q(r)} \right\rangle$$

$q^{th}$ LDOS moment encoded by an operator $\mathcal{O}_q(r)$ in the NL$\sigma$M

$$\nu(r) = \sum_\alpha |\psi_\alpha(r)|^2 \delta(\varepsilon - \varepsilon_\alpha); \quad \nu^q(r) \sim \mathcal{O}_q(r)$$

At the critical point describing the MIT, the expectation of $q^{th}$ LDOS moment scales with system size according to

$$\left\langle \mathcal{O}_q(r) \right\rangle \sim \frac{1}{L^{x_q}}$$

$$x_q = -\sqrt{\frac{e}{2}}q(q-1) + \ldots$$

Wegner 1980, 1987; Pruisken 1985