Termination of typical wavefunction multifractal spectra at the Anderson metal-insulator transition

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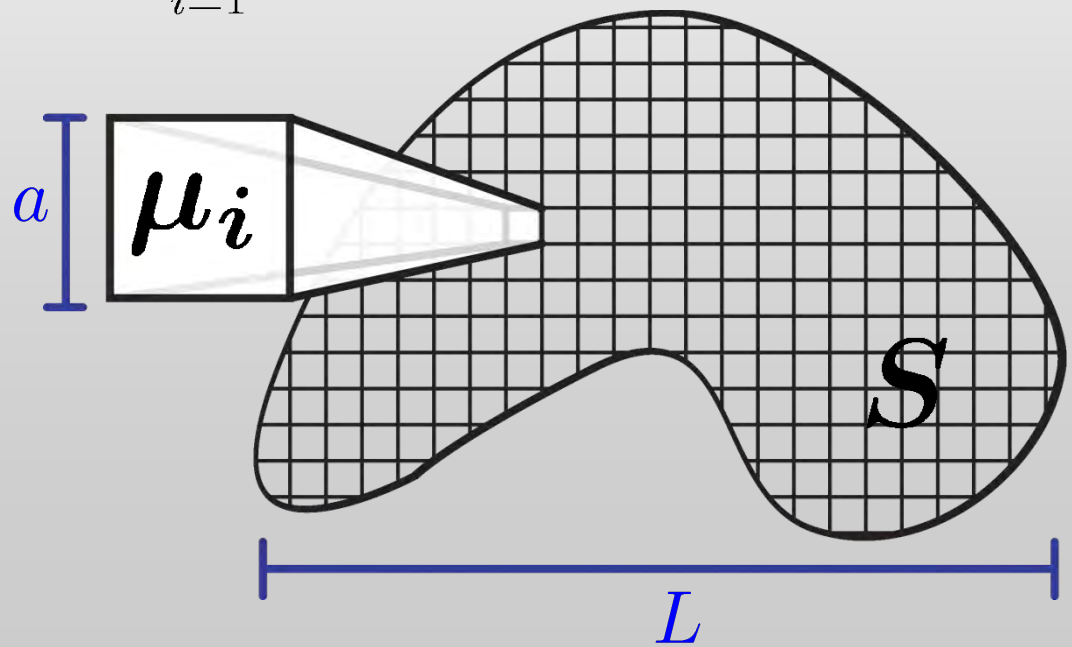
Multifractal: An object characterized by a continuous spectrum of fractal dimensions

- Set S of dimension d, broken up into N pieces ("boxes") Ω_i

  Box size $a \ll$ system size L; $N = (L/a)^d$

- Some probability measure $d\mu$

- <u>Box probability</u>: $\mu_i \equiv \int_{\Omega_i} d\mu; \quad \sum_{i=1}^{N} \mu_i = 1$



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Anderson localization physics (non-interacting electrons):

$$d\mu = |\psi(\mathbf{r})|^2 \, d^d\mathbf{r}; \qquad \left[-\frac{\nabla^2}{2m} + v(\mathbf{r})\right] \psi(\mathbf{r}) = \varepsilon \, \psi(\mathbf{r})$$

Inverse Participation Ratio (IPR): $P_q = \sum_{i=1}^{N} \mu_i^q \sim \left(\frac{a}{L}\right)^{\tau(q)}$

The multifractal spectrum $\tau(q)$: localized versus extended behavior

Inverse Participation Ratio (IPR):

$$P_q \sim \int_{L^d} |\psi(\mathbf{r})|^{2q}\, d^d\mathbf{r} \sim \left(\frac{a}{L}\right)^{\tau(q)}$$

- **Compute for *typical* ψ in a system of size L^d, obtained in a fixed realization of the disorder potential (i.e. numerics)**

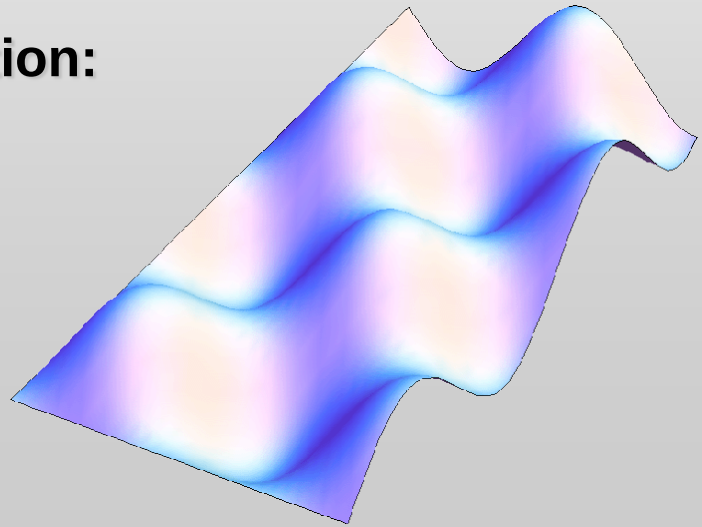
- **Analytical properties:**

$$\frac{d\tau(q)}{dq} \geq 0; \qquad \frac{d^2\tau(q)}{dq^2} \leq 0; \qquad \lim_{q\to\pm\infty} \frac{\tau(q)}{q-1} = d_{\pm\infty}$$

- **As a probe of wavefunction (de)localization:**

(a) Extended plane wave state

$$\tau(q) = d(q-1)$$



The multifractal spectrum $\tau(q)$: localized versus extended behavior

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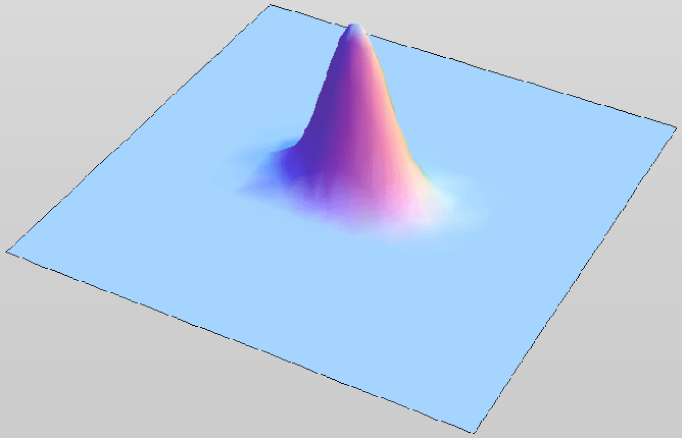
- **As a probe of wavefunction (de)localization:**

(b) Exponentially localized state

$$\tau(q) \sim 0, \qquad L \gg \xi_{\mathrm{loc}}$$

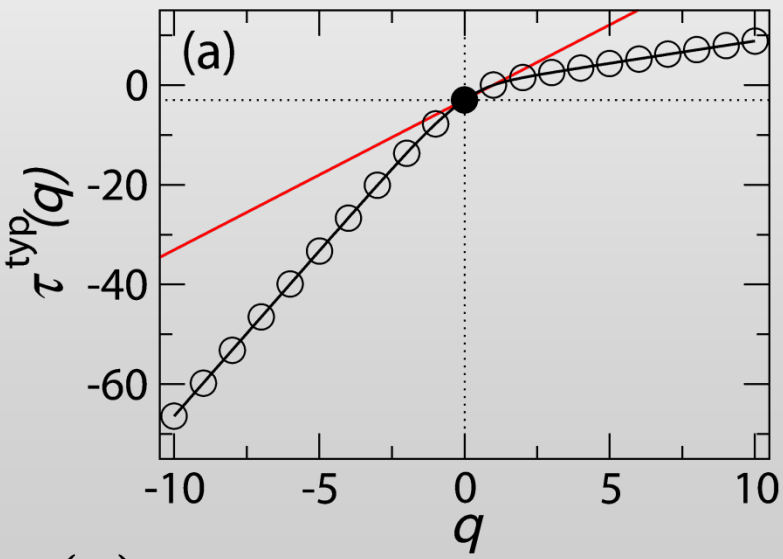


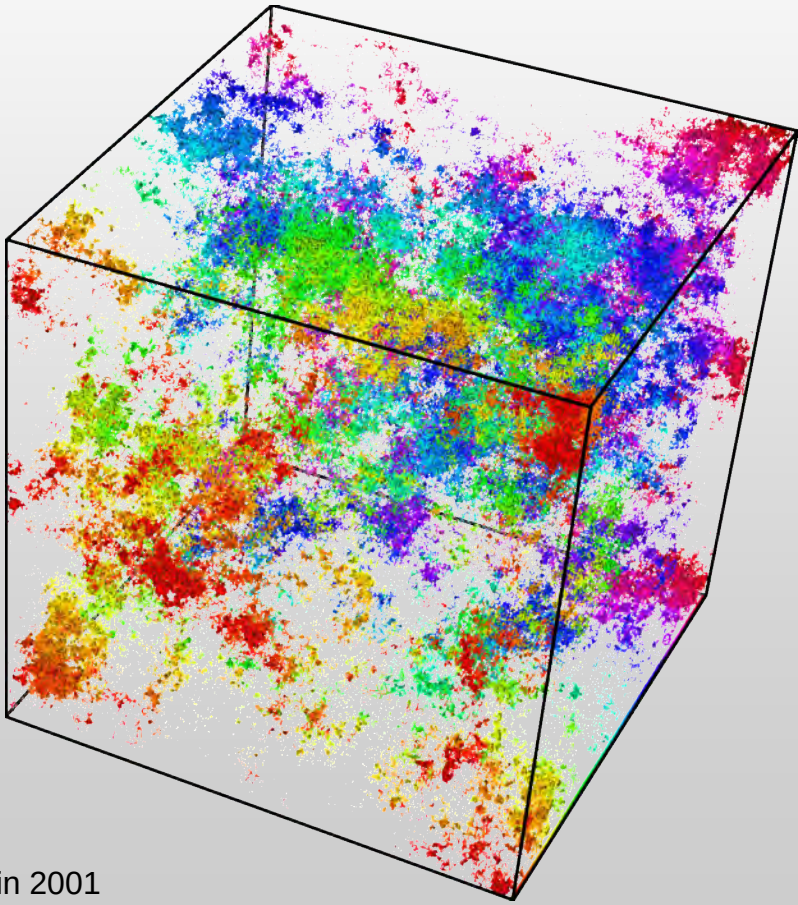
Multifractal spectrum: *Fingerprint* of the MIT

Critical wavefunctions at a delocalization transition
(Anderson MIT in d > 2, IQHP in d = 2)

- ψ **exhibits *multifractal scaling***
  (neither localized nor extended, nor a simple fractal)
  Wegner 1980; Castellani and Peliti 1986

- **Multifractality encoded in *nonlinearity* of the $\tau(q)$ spectrum**





- $\tau(q)$ **is <u>self-averaging</u> and <u>universal</u>**
  e.g., Pook, Janssen 1991; Huckestein 1995; Evers, Mildenberger, Mirlin 2001
  Chamon, Mudry, Wen 1996; Mirlin and Evers 2000

Vasquez, Rodriguez, Roemer 2008

The Anderson MIT in *d* > 2

- **"All states are localized in d = 2"**
- **Mobility edge/MIT in d > 2**
- **MIT is continuous (2nd order) QPT, critical conductance $G = G^*$, independent of L**

  Abrahams, Anderson, Licciardello, Ramakrishnan 1979
- **Average local density of states (LDOS) is non-critical**

  Wegner 1981, McKane and Stone 1981
- **Conventional analytical framework: ϵ-expansion about $d = 2 + \epsilon$, using the non-linear sigma model (NLσM) effective field theory**

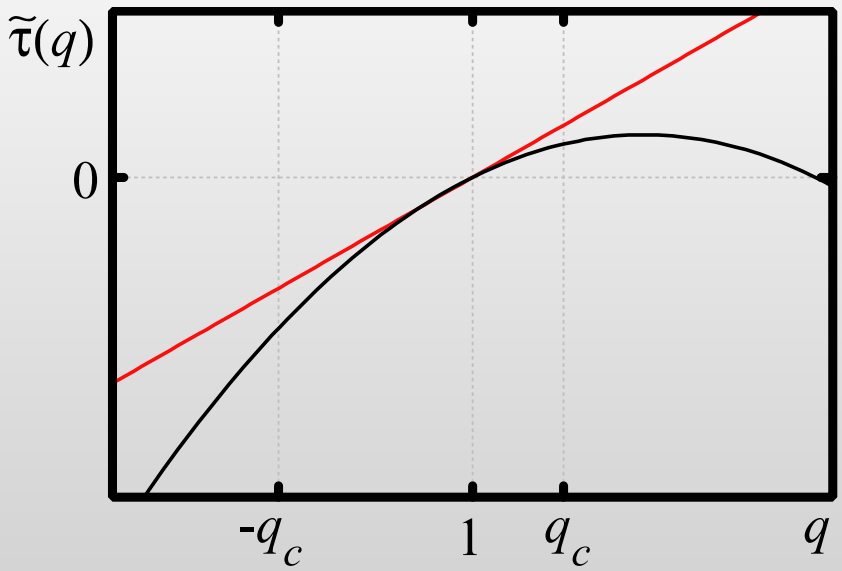
  Wegner 1979

Average vs. Typical Multifractal Spectra

Standard result for the unitary class (disordered metal with broken TRI):

$$\tilde{\tau}(q) = d(q-1) - \sqrt{\frac{\epsilon}{2}}q(q-1) + \boldsymbol{O}\left[\epsilon^2 q^2 (q-1)^2\right]$$

Wegner 1980, 1987; Pruisken 1985



Something is rotten about this result:

- $\tilde{\tau}(q)$ **is not monotonically increasing**
- $\tilde{\tau}(q)$ **becomes NEGATIVE for large, positive** $q \gg 1$

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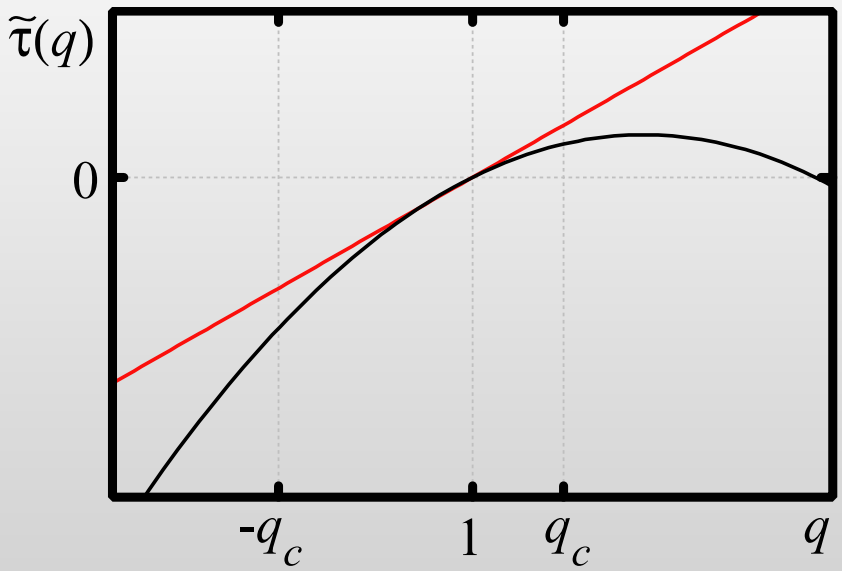

We must take a closer look at the NLσM method:

Typical **spectrum defined for representative wavefunction in a fixed disorder realization**

Analyical calculation: Average over disorder realizations!

$$\tilde{\tau}(q) = -\frac{d \ln \overline{P_q}}{d \ln L}$$

Average spectrum

$$\tau(q) = -\frac{d \overline{\ln P_q}}{d \ln L}$$

Typical spectrum

Extracting the typical multifractal spectrum from the LDOS

IPR in terms of the local density of states (LDOS):

$$P_q = \int |\psi(\mathbf{r})|^{2q}\, d^d\mathbf{r} \sim \frac{\int \nu^q(\mathbf{r})\, d^d\mathbf{r}}{\left[\int \nu(\mathbf{r})\, d^d\mathbf{r}\right]^q}$$

where

$$\nu(\mathbf{r}) = \sum_\alpha |\psi_\alpha(\mathbf{r})|^2\, \delta(\varepsilon - \varepsilon_\alpha)$$

Cumulant function for the *q*th moment of the *<u>integrated</u>* LDOS:

$$F_q(\xi; L) \equiv \left\langle e^{-\xi \int d^d\mathbf{r}\, \nu^q(\mathbf{r})} \right\rangle$$

Extracting the typical multifractal spectrum from the LDOS

- **Typical spectrum:**

$$\tau(q) = \frac{d}{d\ln L}\int_0^\infty \frac{d\xi}{\xi}\left[F_q(\xi;L) - qF_1(\xi;L)\right]$$

- **Average spectrum:**

$$\tilde{\tau}(q) = d(q-1) + x_q - q\,x_1$$

where x_q **is the** ***scaling dimension*** **of** $\nu^q(\mathbf{r})$ **at the MIT in** $d = 2+\epsilon$ **:**

$$\langle \nu^q(\mathbf{r})\rangle \sim \frac{1}{L^{x_q}}$$

Cumulant function for the *q*th moment of the *integrated* LDOS:

$$F_q(\xi;L) \equiv \left\langle e^{-\xi\int d^d\mathbf{r}\,\nu^q(\mathbf{r})}\right\rangle$$

Effective low energy field theory: the NLσM

Wegner 1979; Efetov, Larkin, Khmelnitskii 1980

$$Z = \int \mathcal{D}[\hat{Q}] e^{-S}$$

$$S[\hat{Q}] = \frac{1}{2t} \int d^d\mathbf{r}\, \mathrm{Tr}\left(\boldsymbol{\nabla}\hat{Q} \cdot \boldsymbol{\nabla}\hat{Q}\right) - h \int d^d\mathbf{r}\, \nu(\mathbf{r})$$

where $\hat{Q}(\mathbf{r})$ is an 2n x 2n matrix field satisfying

$$\hat{Q}^\dagger = \hat{Q},\ \hat{Q}^2 = \hat{1},\ \mathrm{Tr}(\hat{Q}) = 0$$

<u>Fermionic</u> <u>replicas</u>: $\hat{Q} \in \dfrac{G(2n)}{G(n) \times G(n)},\ G \in \mathrm{U},\ \mathrm{Sp},\ \mathrm{O}$

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<u>Parameterization: "π - σ" coordinates</u>:

$$\hat{Q} = \begin{bmatrix} (\hat{1} - \hat{W}\hat{W}^\dagger)^{1/2} & \hat{W} \\ \hat{W}^\dagger & -(\hat{1} - \hat{W}^\dagger\hat{W})^{1/2} \end{bmatrix},$$

$\hat{W}(\mathbf{r})$ **is n x n complex (unconstrained)**

The LDOS in the NLσM

The local density of states is represented by the composite operator

$$\nu(\mathbf{r}) \sim \mathrm{Tr}\left(\begin{bmatrix}\hat{1} & 0\\ 0 & -\hat{1}\end{bmatrix}\begin{bmatrix}\hat{Q}_{11} & \hat{Q}_{12}\\ \hat{Q}_{21} & \hat{Q}_{22}\end{bmatrix}\right) = \mathrm{Tr}(\hat{Q}_{11} - \hat{Q}_{22})$$

Consider the case of a fixed number of replicas $n \in \{1, 2, 3, \ldots\}$,

- **2nd order critical point in** $d = 2 + \epsilon$ **at** $t = t_n^*,\ h = 0$
- **symmetry between "π"** $(\hat{W},\ \hat{W}^\dagger)$ **and "σ"** $(\sqrt{\hat{1} - \hat{W}\hat{W}^\dagger},\ \sqrt{\hat{1} - \hat{W}^\dagger\hat{W}})$ **components restored at the transition**

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➡ **At the MIT, we are free to rotate**

$$\nu(\mathbf{r}) \to \mathrm{Tr}\left(\begin{bmatrix}0 & \hat{1}\\ \hat{1} & 0\end{bmatrix}\begin{bmatrix}\hat{Q}_{11} & \hat{Q}_{12}\\ \hat{Q}_{21} & \hat{Q}_{22}\end{bmatrix}\right) = \mathrm{Tr}(\hat{W} + \hat{W}^\dagger)$$

LDOS and its moments in the NLσM

LDOS Moments

$$\nu^q(\mathbf{r}) \sim \left[\mathrm{Tr}(\hat{W} + \hat{W}^\dagger)\right]^q$$

Problem: **Not an eigenoperator; want most relevant component (for a given, fixed q, at the MIT critical point in $d = 2 + \epsilon$)**

Consider the case n = 1: $\dfrac{U(2)}{U(1) \times U(1)} \sim \dfrac{O(3)}{O(2)}$

$$\hat{W} = \pi,\ \hat{W}^\dagger = \pi^*,\ \sigma = (1 - \pi^*\pi)^{1/2}$$

Eigenoperators: $Y_{l,m}(\pi, \pi^*, \sigma), \quad x_l = \dfrac{\epsilon}{2} l(l+1)$ **(at the critical pt in $d = 2 + \epsilon$)**

Brezin, Zinn-Justin, Le Guillou 1976

LDOS and its moments in the NLσM

(n = 1, continued…)

<u>Eigenoperators</u>: $Y_{l,m}(\pi, \pi^*, \sigma), \quad x_l = \frac{\epsilon}{2} l(l+1)$ **[at the critical pt in** $d = 2 + \epsilon$ **]**

$$\Rightarrow \nu^q \sim (\pi + \pi^*)^q = \sum_{l=0}^{q} \mathcal{O}_l^{(q)}$$

Addition of angular momentum: $\mathcal{O}_l^{(q)} = \sum_{m=-l}^{l} \kappa_{l,m}^{(q)} Y_{l,m}(\pi, \pi^*, \sigma)$

LDOS and its moments in the NLσM

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In particular, $\mathcal{O}_q^{(q)} = (\pi^q + \ldots + \pi^{*\,q})$

Lowest, highest weight states $Y_{q,\mp q}$

$$\langle \mathcal{O}_q^{(q)} \rangle \sim \frac{1}{L^{x_q}}, \; x_q = \frac{\epsilon}{2} q(q+1)$$

Largest positive (*least* relevant) scaling dimension contributing to $\langle \nu^q \rangle$

LDOS and its moments in the NLσM

(n = 1, continued…)

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OPE at ZERO coupling (t = 0):

$$\mathcal{O}_q^{(q)}(\mathbf{r})\,\mathcal{O}_{q'}^{(q')}(\mathbf{r'}) = C_{q,q'}^{q+q'}\,\mathcal{O}_{q+q'}^{(q+q')}[(\mathbf{r}+\mathbf{r'})/2] + \ldots$$

$$(Y_{q,-q} + \ldots)(Y_{q',-q'} + \ldots) = C_{q,q'}^{q+q'}\,(Y_{q+q',-q-q'} + \ldots)$$

LDOS and its moments in the NLσM

<u>Replica</u> <u>NLσM</u>: (general n, n → 0)

$$\nu^q(\mathbf{r}) \sim \left[\mathrm{Tr}(\hat{W} + \hat{W}^\dagger)\right]^q$$

Lowest weight state in most relevant representation?

$$\left[\mathrm{Tr}\left(\hat{W}\right)\right]^q$$ ➡ **No—still not an eigenoperator**

Altshuler, Kratsov, Lerner 1986, 1989

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Most relevant component:

$$\mathcal{O}_q(\mathbf{r}) \equiv \mathcal{O}^{\alpha_1\alpha_2\ldots\alpha_q}_{q\,[\alpha_1\alpha_2\ldots\alpha_q]}(\mathbf{r}),\ x_q = -\sqrt{\frac{\epsilon}{2}}q(q-1) + \boldsymbol{O}\left[\epsilon^2 q^2 (q-1)^2\right]$$

where

$$\mathcal{O}^{\alpha_1\alpha_2\ldots\alpha_q}_{q\,[\beta_1\beta_2\ldots\beta_q]}(\mathbf{r}) \equiv \left(\frac{1}{q!}\right)^2 \sum_{\mathbf{P}} \mathrm{sgn}(\mathbf{P})\ \left[W^{\alpha_1}{}_{\beta_{\mathbf{P}(1)}} \cdots W^{\alpha_q}{}_{\beta_{\mathbf{P}(q)}}\right]$$

Wegner 1980, 1987; Pruisken 1985

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Most relevant component:

$$\mathcal{O}_2 = \frac{\left[\mathrm{Tr}\left(\hat{W}\right)\right]^2 - \mathrm{Tr}\left(\hat{W}^2\right)}{(2!)^2}$$

$$\mathcal{O}_3 = \frac{\left[\mathrm{Tr}\left(\hat{W}\right)\right]^3 - 3\,\mathrm{Tr}\left(\hat{W}^2\right)\mathrm{Tr}\left(\hat{W}\right) + 2\,\mathrm{Tr}\left(\hat{W}^3\right)}{(3!)^2}$$

Summary: LDOS and its moments in the NLσM

At the critical point describing the MIT, the expectation of *q*th LDOS moment scales with system size according to

$$\langle \mathcal{O}_q(\mathbf{r}) \rangle \sim \frac{1}{L^{x_q}}, \quad x_q = -\sqrt{\frac{\epsilon}{2}} q(q-1) + \ldots$$

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Key points: LDOS moment exponents (scaling dimensions)

1) x_q always <u>negative</u> for q>1—LDOS moment operators always relevant perturbations (in the RG sense, at the MIT)

$$x_q < 0, \quad q > 1$$

2) Convex property: higher moments <u>more relevant</u> than lower ones!

$$x_{q+q'} < x_q + x_{q'} < 0$$

…back to the multifractal spectrum

$$\langle \mathcal{O}_q(\mathbf{r}) \rangle \sim \frac{1}{L^{x_q}}, \quad x_q = -\sqrt{\frac{\epsilon}{2}} q(q-1) + \ldots$$

- **Typical spectrum:**

$$\tau(q) = \frac{d}{d \ln L} \int_0^\infty \frac{d\xi}{\xi} \left[F_q(\xi; L) - q F_1(\xi; L) \right]$$

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Cumulant function for the qth moment of the *integrated* LDOS:

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The cumulant expansion is Doomed!

Simplest approach: cumulant expansion

$$F_q(\xi; L) \sim e^{-\xi \int d^d\mathbf{r}\, \langle \mathcal{O}_q(\mathbf{r})\rangle + \frac{\xi^2}{2!}\left[\int d^d\mathbf{r} d^d\mathbf{r'} \left\langle \mathcal{O}_q(\mathbf{r})\mathcal{O}_q(\mathbf{r'})\right\rangle_c\right] + \ldots}$$

- **First term: average *is* typical** $\tilde{\tau}(q) = \tau(q)$
- **Second term: Operator Product Expansion (OPE) means TROUBLE!**

$$\mathcal{O}_q(\mathbf{r})\,\mathcal{O}_{q'}(\mathbf{r'}) \sim \frac{C_{q,q'}^{q+q'}}{|\mathbf{r}-\mathbf{r'}|^{x_q + x_{q'} - x_{q+q'}}} \mathcal{O}_{q+q'}\left(\frac{\mathbf{r}+\mathbf{r'}}{2}\right) + \ldots$$

$$x_q < 0, \quad q > 1$$
$$x_{q+q'} < x_q + x_{q'} < 0$$

Higher cumulants give ever LARGER contributions! Cumulant expansion does not converge!

Solution: the Functional Renormalization Group (FRG)

Following Carpentier and Le Doussal (2000) and Mudry, Ryu, Furusaki (2003), we use the FRG to keep track of all moments:

1. Generalize F_q, including all higher moments from very beginning

$$F_q(\xi; L) = \left\langle e^{\sum_{p=1}^{\infty} y_{pq} \int d^d\mathbf{r}\, \mathcal{O}_{pq}(\mathbf{r})} \right\rangle; \quad y_{pq}(0) = -\xi\delta_{p,1}$$

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2. Study scaling of F_q: write RG flow equations for coupling constants $\{y_{pq}\}$ *at the MIT in* $d = 2 + \epsilon$ *(using the OPE)*

$$\frac{d\, y_{pq}}{dl} = (d - x_q) y_{pq} + \frac{S_d}{2} \sum_{m=1}^{p-1} C^p_{m,p-m}\, y_{mq}\, y_{(p-m)q} + \boldsymbol{O}\left(y^3\right)$$

where

$$x_q = -\sqrt{\frac{\epsilon}{2}} q(q-1) + \boldsymbol{O}\left[\epsilon^2 q^2 (q-1)^2\right]$$

Wegner 1980, 1987; Pruisken 1985

$$C^{q+q'}_{q,q'} = \binom{q+q'}{q} \left[1 + \boldsymbol{O}(\epsilon)\right]$$

NEW Foster, Ryu, Ludwig 2009

Solution: the Functional Renormalization Group (FRG)

3. Write auxillary generating functional for the coupling constants

$$G_q(z,l) \equiv \tilde{G}_q(\tilde{z},\tilde{l}) \equiv 1 + \frac{S_d}{2d}\sum_{p=1}^{\infty}\frac{(e^{-z})^p}{p!}y_{pq}(l)$$

Initial condition:

$$G_q(z,0) \sim \exp\left[-\xi\frac{S_d}{2d}e^{-z}\right]$$

Galilean boost:

$$\tilde{z} = z + \sqrt{\frac{\epsilon}{2}}q\,l,\ \tilde{l} = l$$

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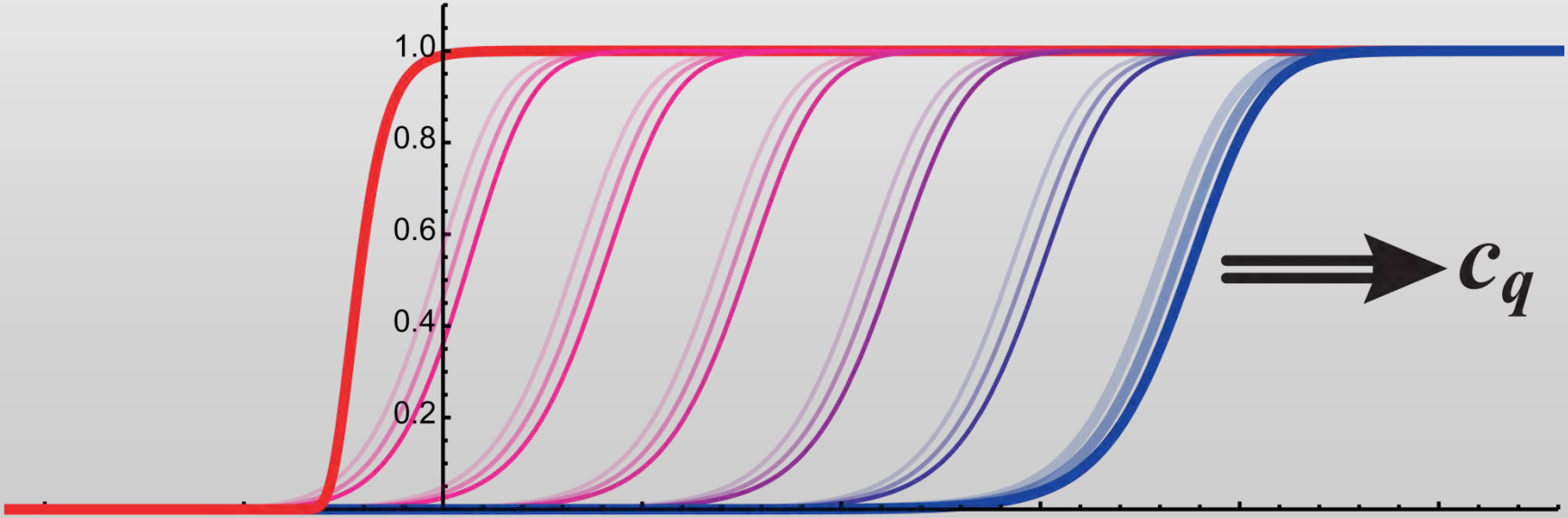
4. $\tilde{G}_q$ **satisfies the Kolmogorov-Petrovsky-Piscounov (KPP) equation, which describes non-linear diffusion in 1D.**

$$\left[\frac{1}{d}\partial_{\tilde{l}} - D_q\partial^2_{\tilde{z}}\right]\tilde{G}_q = \tilde{G}_q(\tilde{G}_q - 1); \quad D_q = \frac{q^2}{d}\sqrt{\frac{\epsilon}{2}}$$

Solution: the Functional Renormalization Group (FRG)

KPP has propagating front solutions, with a *universal front velocity* $\tilde{c}_q$.
Velocity selection: $\tilde{c}_q$ is a non-analytic function of the diffusion constant

$$\tilde{G}_q(\tilde{z},\tilde{l})|_{l\to\infty} \sim h(\tilde{z}-\tilde{c}_q\tilde{l});\;\; \tilde{c}_q = \begin{cases} d\,(1+D_q)\,, & D_q \le 1 \\ 2d\sqrt{D_q}, & D_q > 1 \end{cases}$$



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KPP has propagating front solutions, with a *universal front velocity* $\tilde{c}_q$.
Velocity selection: $\tilde{c}_q$ is a non-analytic function of the diffusion constant

$$\tilde{G}_q(\tilde{z},\tilde{l})|_{l\to\infty} \sim h(\tilde{z}-\tilde{c}_q\tilde{l}); \;\; \tilde{c}_q = \begin{cases} d\,(1+D_q)\,, & D_q \le 1 \\ 2d\sqrt{D_q}, & D_q > 1 \end{cases}$$

G_q **tracks scaling of the coupling constants.**
FRG: the set $\{y_{pq}\}$ "fuses" into single, *typical* coupling $y_q^{\text{typ}} \sim e^{c_q l}$

Associated *typical* scaling dimension:

$$x_q^{\text{typ}} \equiv d - c_q$$
$$= \begin{cases} -\dfrac{d}{q_c^2}q(q-1), & 1 \le q \le q_c \\ d(1-q) + qd\left(1-\dfrac{\text{sgn}(q)}{q_c}\right)^2, & q > q_c \end{cases}$$

$$q_c^2 = 2\sqrt{\frac{2}{\epsilon}}$$

Final result: Termination of the multifractal spectrum at the MIT

Evaluate $F_q(\xi; L)$ in the cumulant expansion, *after* coarse-graining (running the RG)

$$F_q(\xi; L) \sim e^{-y_q^{\text{typ}} \int d^d\mathbf{r} \left\langle \mathcal{O}_q^{\text{typ}}(\mathbf{r}) \right\rangle} \sim e^{-\xi L^{d - x_q^{\text{typ}}}}$$

$$\Rightarrow \quad \tau(q) = \frac{d}{d \ln L} \int_0^\infty \frac{d\xi}{\xi} \left[F_q(\xi; L) - q F_1(\xi; L) \right]$$

$$= d(q-1) + x_q^{\text{typ}} - q\, x_1^{\text{typ}}$$

KPP velocity selection = Termination of the typical multifractal spectrum

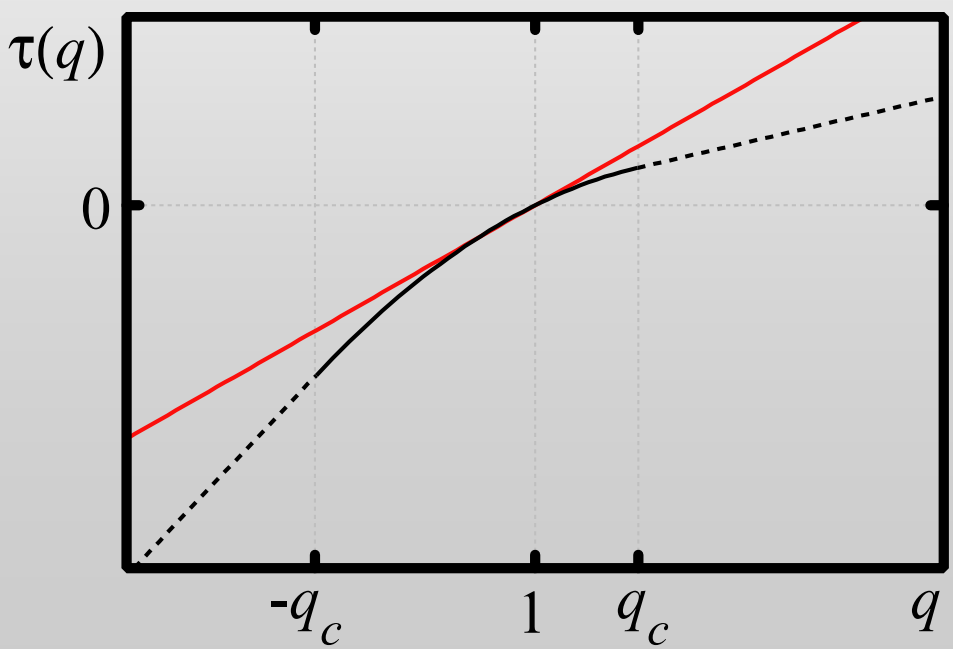
$$\tau(q) = \begin{cases} \tilde{\tau}(q) = d(q-1)\left(1 - \frac{q}{q_c^2}\right), & |q| < q_c \\ d\left(1 - \frac{\text{sgn}(q)}{q_c}\right)^2 q, & |q| > q_c \end{cases}$$

Final result: Termination of the multifractal spectrum at the MIT

KPP velocity selection = Termination of the typical multifractal spectrum

$$\tau(q) = \begin{cases} \tilde{\tau}(q) = d(q-1)\left(1 - \frac{q}{q_c^2}\right), & |q| < q_c \\ d\left(1 - \frac{\mathrm{sgn}(q)}{q_c}\right)^2 q, & |q| > q_c \end{cases}$$

$$q_c^2 \sim 2\sqrt{\frac{2}{\epsilon}}$$



Interpretation:

IPR for representative wavefunction, *typical* disorder realization, dominated by rare wavefunction extrema for large |q|--simple fractal behavior beyond termination.

Validity of termination within the epsilon expansion:

$$\tau(q) = \begin{cases} \tilde{\tau}(q) = d(q-1)\left(1 - \frac{q}{q_c^2}\right), & |q| < q_c \\ d\left(1 - \frac{\mathrm{sgn}(q)}{q_c}\right)^2 q, & |q| > q_c \end{cases}$$

$$q_c^2 \sim 2\sqrt{\frac{2}{\epsilon}}$$

4-loop result for the scaling dimension (Wegner 1987):

$$x_q = -\sqrt{\frac{\epsilon}{2}}q(q-1) - \frac{3\zeta(3)}{8}\epsilon^2 q^2 (q-1)^2 + \boldsymbol{O}\left(\epsilon^{5/2}\right)$$

At the termination threshold:

$$x_{q=q_c} = -2 + (2\epsilon)^{1/4} + \boldsymbol{O}\left(\epsilon^{1/2}\right)$$

Speculative extrapolation…

Although uncontrolled, it is interesting to consider the large-ϵ limit of the FRG predictions: (<u>strong</u> <u>disorder</u>, <u>high</u> <u>dimensionality</u>)

1. **For $\epsilon > \epsilon_{\mathsf{F}} \sim 8$, $q_c < 1$. Then the *typical* LDOS acquires a non-zero scaling dimension**

$$x_1^{\mathsf{typ}} = d\left(1 - \frac{1}{q_c}\right)^2 > 0$$

?? Compare to Bethe lattice $(d \to \infty)$, where typical LDOS vanishes *exponentially* across the transition.

Abou-Chacra, Anderson, Thouless, 1973;
Mirlin and Fyodorov 1994; Dobrosavljevic and Kotliar 1997

2. **Typical multifractal spectrum exhibits *freezing*:**

Mudry, Chamon, Wen 1996;
Carpentier and Le Doussal 2001

$$\tau(q) = \begin{cases} -d\left(1 - \dfrac{q}{q_c}\right)^2, & |q| \leq q_c \\ \dfrac{2d}{q_c}\left(q - |q|\right). & |q| > q_c \end{cases}$$

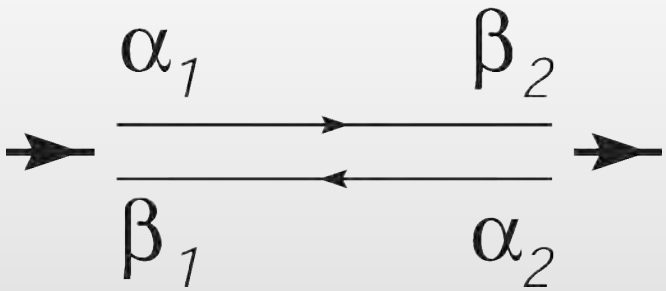
$$\therefore \tau(q) = 0$$

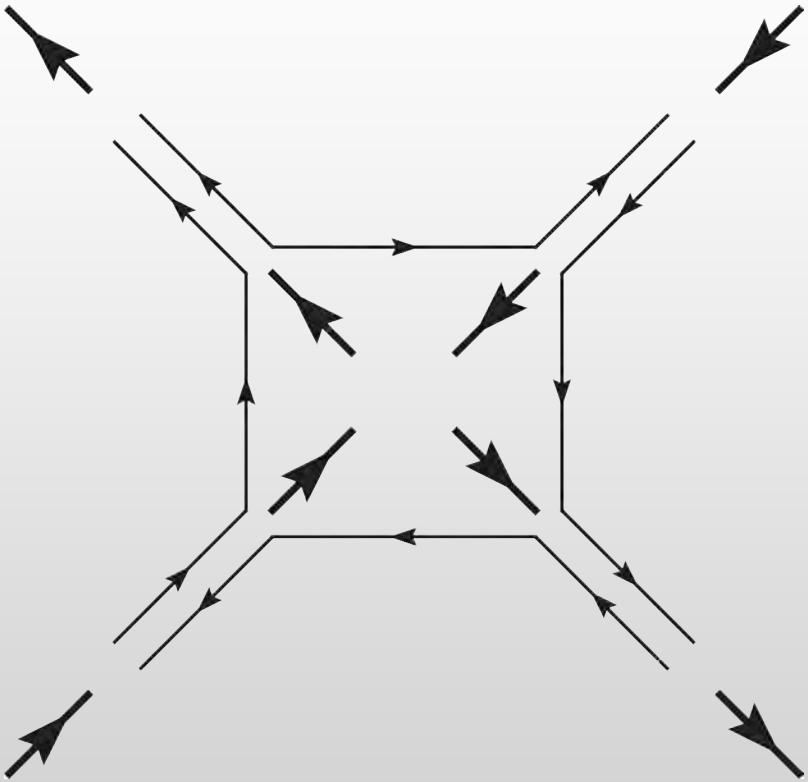
$$q \in \{1, 2, 3, \ldots\}$$

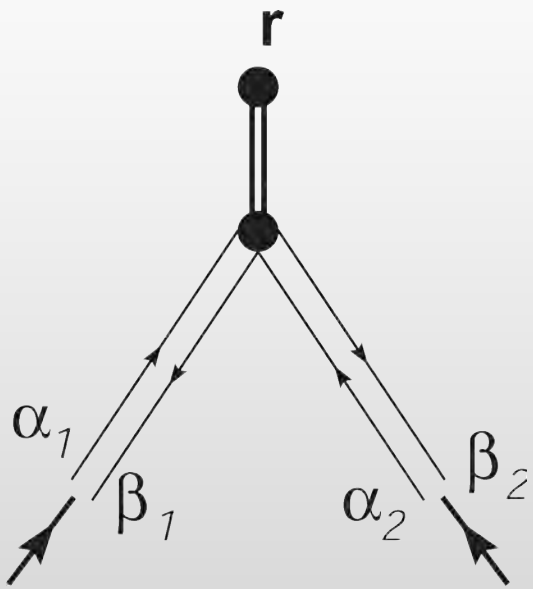
Summary

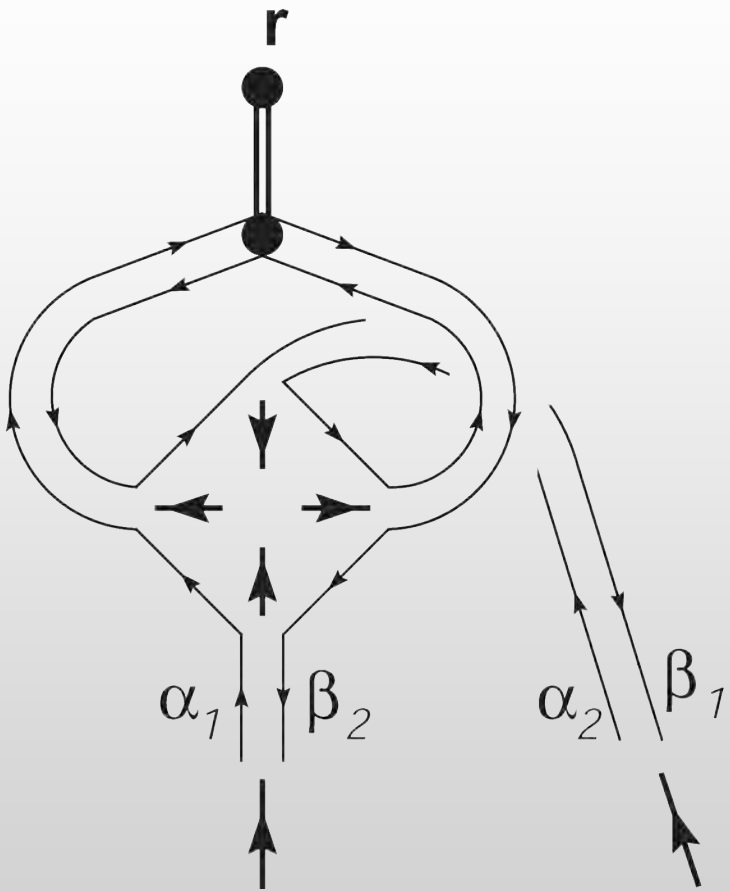
- At a random critical point, entire distribution functions often required to characterize observables: Functional RG method is required
- Typical multifractal spectrum should be <u>universal</u> and <u>self-averaging</u>

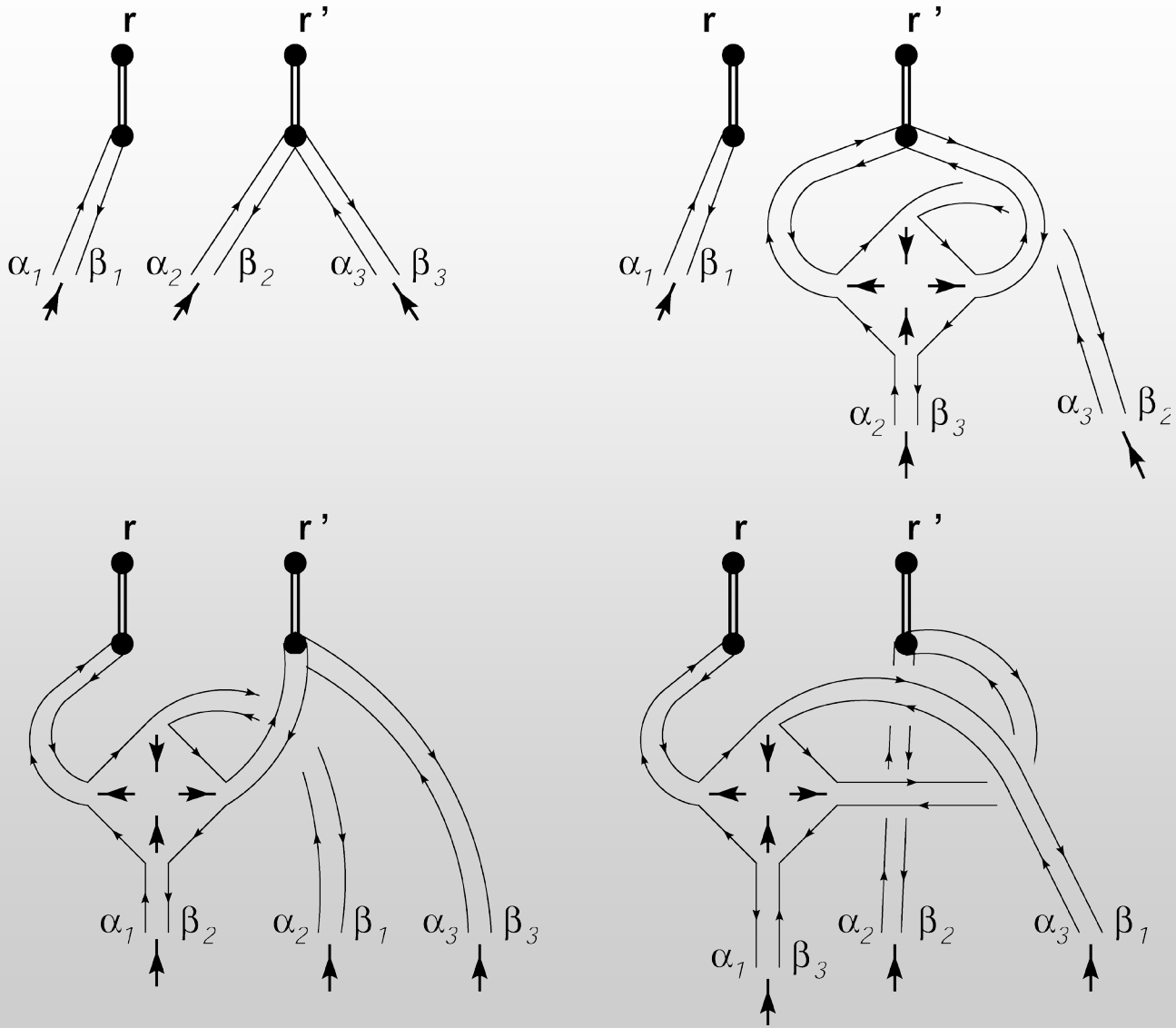
  - Extract from FRG for the LDOS moment generating function
  - Key input: the OPE between different moment operators
  - Mapping of FRG flow equations to KPP
  - KPP velocity selection determines termination of the typical spectrum
- Interesting (speculative) connections to high-*d* Bethe lattice results (see paper for details)


α_1
β_2
β_1
α_2




r
α1
β1
α2
β2


r
α1
β2
α2
β1


r
r'
α1
β1
α2
β2
α3
β3
r
r'
α1
β1
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β3
α3
β2
r
r'
α1
β2
α2
β1
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β3
r
r'
α1
β3
α2
β2
α3
β1

How to extract the Typical multifractal spectrum

Typical spectrum obtains via the LDOS Moment Generating Function F_q :

$$\tau(q) = \frac{d}{d\ln L}\int_0^\infty \frac{d\xi}{\xi}\left[F_q(\xi;L) - qF_1(\xi;L)\right]$$

$$F_q(\xi;L) = \left\langle e^{-\xi\int d^d\mathbf{r}\,\mathcal{O}_q(\mathbf{r})}\right\rangle$$

***q^th* LDOS moment encoded by an operator $\mathcal{O}_q(\mathbf{r})$ in the NLσM**

$$\nu(\mathbf{r}) = \sum_\alpha |\psi_\alpha(\mathbf{r})|^2\,\delta(\varepsilon-\varepsilon_\alpha); \quad \nu^q(\mathbf{r}) \sim \mathcal{O}_q(\mathbf{r})$$

At the critical point describing the MIT, the expectation of *q^th* LDOS moment scales with system size according to

$$\langle\mathcal{O}_q(\mathbf{r})\rangle \sim \frac{1}{L^{x_q}} \qquad x_q = -\sqrt{\frac{\epsilon}{2}}q(q-1) + \ldots$$

Wegner 1980, 1987; Pruisken 1985