Termination of Typical Wavefunction Multifractal Spectra at the Anderson MIT

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Multifractal Spectrum and Localization

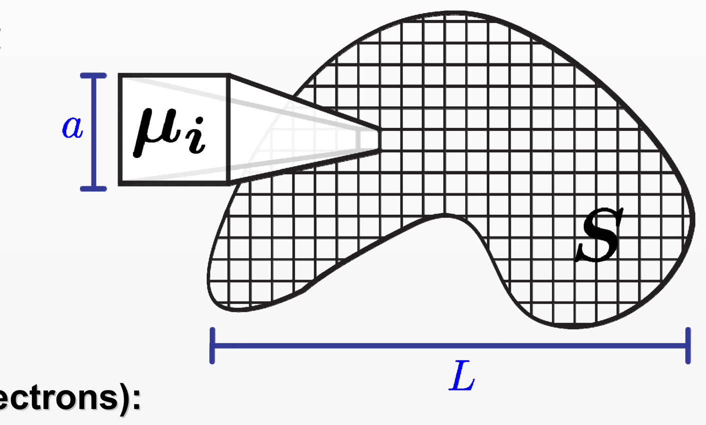
Multifractal: An object characterized by a continuous spectrum of fractal dimensions

- Set S of dimension d, broken up into N pieces ("boxes") Ω_i

  Box size $a \ll$ system size L; $N = (L/a)^d$

- Some probability measure $d\mu$

- Box probability: $\mu_i \equiv \int_{\Omega_i} d\mu; \quad \sum_{i=1}^{N} \mu_i = 1$



Anderson localization physics (non-interacting electrons):

$$d\mu = |\psi(\mathbf{r})|^2 \, d^d\mathbf{r}; \quad \left[-\frac{\nabla^2}{2m} + v(\mathbf{r})\right]\psi(\mathbf{r}) = \varepsilon\,\psi(\mathbf{r}) \quad \left\{ \text{electron moving in a random impurity potential } v(\mathbf{r}) \right\}$$

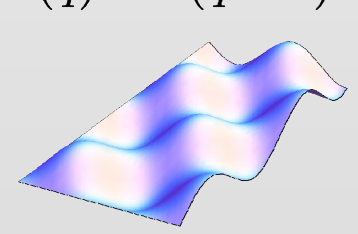
Inverse Participation Ratio (IPR): $P_q \equiv \sum_{i=1}^{N} \mu_i^q \sim \left(\frac{a}{L}\right)^{\tau(q)}$

The multifractal spectrum $\tau(q)$: localized versus extended behavior

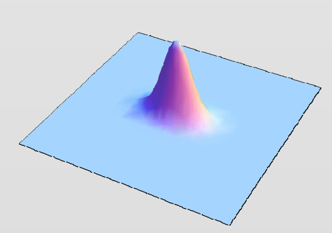
Inverse Participation Ratio (IPR): $P_q \sim \int_{L^d} |\psi(\mathbf{r})|^{2q} \, d^d\mathbf{r} \sim \left(\frac{a}{L}\right)^{\tau(q)}$

- Compute for *typical* ψ in a system of size L^d, obtained in a fixed realization of the disorder potential (i.e. numerics)
- Analytical properties: $\frac{d\tau(q)}{dq} > 0; \quad \frac{d^2\tau(q)}{dq^2} \leq 0; \quad \lim_{q\to\pm\infty} \frac{\tau(q)}{q-1} = d_{\pm\infty}$
- As a probe of wavefunction (de)localization:

(a) Extended plane wave state: $\tau(q) = d(q-1)$



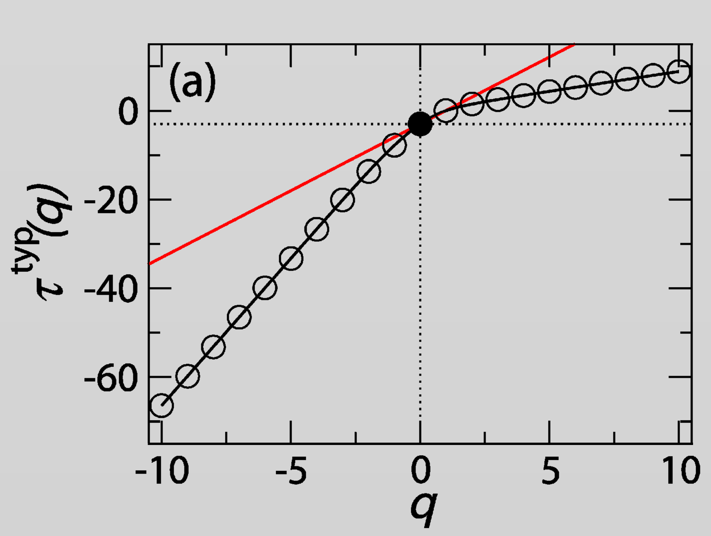
(b) Exponentially localized state: $\tau(q) \sim 0, \quad L \gg \xi_{\text{loc}}$

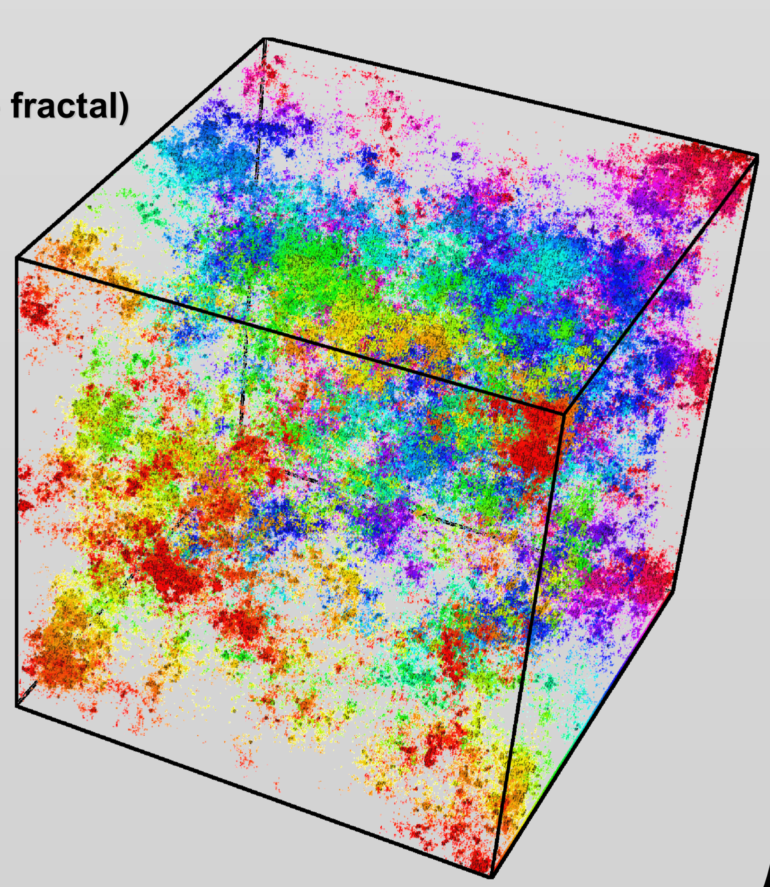


Multifractal spectrum: *Universal Fingerprint* of the MIT

Critical wavefunctions at a delocalization transition
(Anderson MIT in $d > 2$, IQHP in $d = 2$)

- ψ exhibits *multifractal scaling* (neither localized nor extended, nor a simple fractal)
  Wegner 1980; Castellani and Peliti 1986
- Multifractality encoded in *nonlinearity* of the $\tau(q)$ spectrum





- $\tau(q)$ is self-averaging and universal
  e.g., Pook, Janssen 1991; Huckestein 1995; Evers, Mildenberger Mirlin 2001
  Chamon, Mudry, Wen 1996; Mirlin and Evers 2000

Numerical $\tau(q)$ spectrum and 3D plot of the probability density of a typical multifractal wavefunction at the MIT in $d = 3$
Vasquez, Rodriguez, Roemer 2008

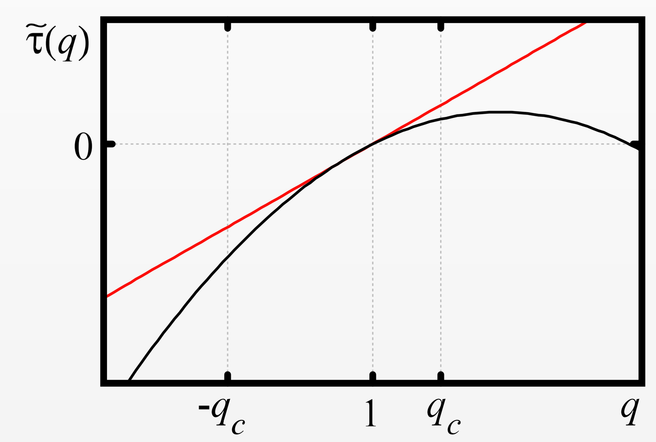
Typical vs Average Spectra at the MIT in $d > 2$

Average vs. Typical Multifractal Spectra

Standard result for the unitary class (disordered metal with broken TRI):

$$\tilde{\tau}(q) = d(q-1) - \sqrt{\frac{\epsilon}{2}}\,q(q-1) + O\left[\epsilon^2 q^2 (q-1)^2\right]$$

Wegner 1980, 1987; Pruisken 1985



Something is **rotten** about this result:

- $\tilde{\tau}(q)$ is not monotonically increasing
- $\tilde{\tau}(q)$ becomes NEGATIVE for large, positive $q \gg 1$

Must distinguish Average vs. Typical spectra!

$$\tilde{\tau}(q) = -\frac{d \ln \overline{(P_q)}}{d \ln L}$$ **Average spectrum**

In prior work (Wegner, Pruisken 1980s), *Average* spectrum obtained by taking the *Log of the disorder-averaged IPR.*

$$\tau(q) = -\frac{d\, \overline{(\ln P_q)}}{d \ln L}$$ **Typical spectrum**

In numerics, *Typical* spectrum extracted from a representative wavefunction in a fixed disorder configuration. Because it is self-averaging, the typical spectrum can also be defined as the *expectation of the Log of the IPR,* averaged over disorder realizations.
Never before computed analytically.

How to extract the Typical multifractal spectrum

Typical spectrum can be extracted from the Moment Generating Function F_q for the q^{th} *moment of local density of states* (LDOS):

$$\tau(q) = \frac{d}{d \ln L} \int_0^\infty \frac{d\xi}{\xi} \left[F_q(\xi; L) - q F_1(\xi; L)\right]; \quad F_q(\xi; L) \equiv \left\langle e^{-\xi \int d^d\mathbf{r}\, \mathcal{O}_q(\mathbf{r})} \right\rangle$$

Here, the q^{th} LDOS moment is encoded by an operator $\mathcal{O}_q(\mathbf{r})$ in the low-energy field theory description of the Anderson localization problem (the "replica NLσM"):

$$\nu(\mathbf{r}) = \sum_\alpha |\psi_\alpha(\mathbf{r})|^2 \, \delta(\varepsilon - \varepsilon_\alpha); \quad \nu^q(\mathbf{r}) \sim \mathcal{O}_q(\mathbf{r})$$

At the critical point describing the MIT (located via the perturbative RG in $d = 2 + \epsilon$), the expectation of q^{th} LDOS moment scales with system size according to

$$\langle \mathcal{O}_q(\mathbf{r}) \rangle \sim \frac{1}{L^{x_q}}, \quad x_q = -\sqrt{\frac{\epsilon}{2}}\,q(q-1) + \ldots$$

Wegner 1980, 1987; Pruisken 1985

The cumulant expansion is Doomed!

Simplest approach to compute the typical $\tau(q)$: Try the cumulant expansion

$$F_q(\xi; L) \sim e^{-\xi \int d^d\mathbf{r}\, \langle \mathcal{O}_q(\mathbf{r}) \rangle + \frac{\xi^2}{2!}\left[\int d^d\mathbf{r}\, d^d\mathbf{r}' \langle \mathcal{O}_q(\mathbf{r}) \mathcal{O}_q(\mathbf{r}') \rangle_c\right] + \ldots}$$

- First term: average *is* typical $\tilde{\tau}(q) = \tau(q)$
- Second term: Operator Product Expansion (OPE) means TROUBLE!

$$\mathcal{O}_q(\mathbf{r})\, \mathcal{O}_{q'}(\mathbf{r}') \sim \frac{C_{q,q'}^{q+q'}}{|\mathbf{r} - \mathbf{r}'|^{x_q + x_{q'} - x_{q+q'}}} \mathcal{O}_{q+q'}\left(\frac{\mathbf{r} + \mathbf{r}'}{2}\right) + \ldots$$

$x_q < 0, \quad q > 1$
$x_{q+q'} < x_q + x_{q'} < 0$

Negative-convex scaling dimension spectrum of the LDOS moments implies that higher cumulants give ever LARGER contributions! Cumulant expansion does not converge!

Typical Spectrum via the FRG

Solution: the Functional Renormalization Group (FRG)

Following Carpentier and Le Doussal (2000) and Mudry, Ryu, Furusaki (2003), we use a Functional Renormalization Group method to keep track of all moments:

1. Generalize F_q, including all higher LDOS moments from the very beginning

$$F_q(\xi; L) = \left\langle e^{\sum_{p=1}^{\infty} Y_{pq} \int d^d\mathbf{r}\, \mathcal{O}_{pq}(\mathbf{r})} \right\rangle; \quad Y_{pq}(0) = -\xi \delta_{p,1}$$

Here $\mathcal{O}_{pq}(\mathbf{r})$ is a local operator representing the $(p \times q)^{th}$ moment of the LDOS.

2. Study scaling of F_q: write RG flow equations for the *coupling constants* $\{Y_{pq}\}$ *at the MIT critical fixed point in* $d = 2 + \epsilon$ (using the OPE)

$$\frac{dY_{pq}}{dl} = (d - x_{pq}) Y_{pq} + \frac{S_d}{2} \sum_{m=1}^{p-1} C_{mq,(p-m)q}^{pq} Y_{mq} Y_{(p-m)q} + O\left(Y^3\right)$$

where

$$x_q = -\sqrt{\frac{\epsilon}{2}}\,q(q-1) + O\left[\epsilon^2 q^2 (q-1)^2\right]$$ Scaling dimension of $\mathcal{O}_q(\mathbf{r})$

Wegner 1980, 1987; Pruisken 1985

$$C_{q,q'}^{q+q'} = \binom{q+q'}{q} \left[1 + O(\epsilon)\right]$$ OPE Coefficient

Foster, Ryu, Ludwig 2009

3. Write an auxiliary generating functional for the coupling constants

$$G_q(z, l) \equiv 1 + \frac{S_d}{2d} \sum_{p=1}^{\infty} \frac{(e^{-z})}{p!} Y_{pq}(l); \quad \tilde{G}_q(\tilde{z}, \tilde{l}) \equiv G_q\left(\tilde{z} - q\tilde{l}\sqrt{\frac{\epsilon}{2}}, \tilde{l}\right)$$

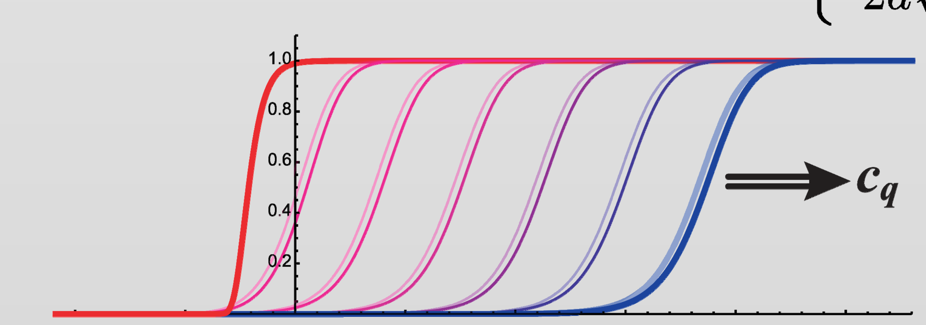
4. $\tilde{G}_q$ satisfies the Kolmogorov-Petrovsky-Piscounov (KPP) equation, which describes non-linear diffusion in 1D.

$$\left[\frac{1}{d}\partial_{\tilde{l}} - D_q \partial_{\tilde{z}}^2\right] \tilde{G}_q = \tilde{G}_q(\tilde{G}_q - 1); \quad D_q = \frac{q^2}{d}\sqrt{\frac{\epsilon}{2}}$$

KPP has propagating front solutions, with a *universal front velocity* $\tilde{c}_q$.

Velocity selection: $\tilde{c}_q$ is a non-analytic function of the diffusion constant

$$\tilde{G}_q(\tilde{z}, \tilde{l})|_{l\to\infty} \sim h(\tilde{z} - \tilde{c}_q \tilde{l}); \quad \tilde{c}_q = \begin{cases} d(1 + D_q), & D_q \leq 1 \\ 2d\sqrt{D_q}, & D_q > 1 \end{cases}$$

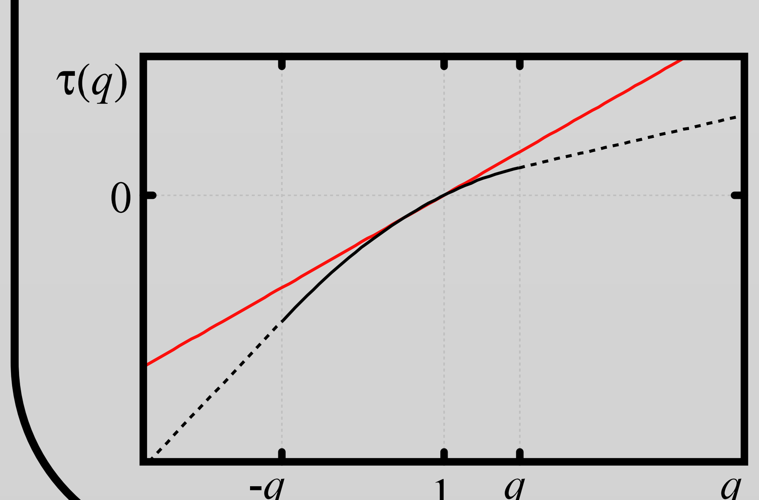


Propagating KPP front solution

Final result: Termination of the multifractal spectrum at the MIT

KPP velocity selection = Termination of the typical multifractal spectrum

$$\tau(q) = \begin{cases} \tilde{\tau}(q) = d(q-1)\left(1 - \frac{q}{q_c^2}\right), & |q| < q_c \\ d\left(1 - \frac{\text{sgn}(q)}{q_c}\right)^2 q, & |q| > q_c \end{cases} \qquad q_c^2 \sim 2\sqrt{\frac{2}{\epsilon}}$$



Physical Interpretation:

The IPR for a representative wavefunction computed in a *typical* disorder realization is dominated by the rare extrema of that wavefunction for large |q|. Simple fractal behavior is found beyond termination, which reflects the *extremal statistics* of the typical wavefunction.

References:
1. Phys. Rev. B **80**, 075101 (2009)
2. T. Vojta, Physics **2**, 66 (2009)

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