A 3D integer quantum Hall effect: Universal spin and heat transport at the surface of a topological superconductor

or:

2D Majorana liquid theory

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February 3rd, 2015
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2D Majorana liquid theory

Hong-Yi Xie, Yang-Zhi Chou, and Matthew S. Foster

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I. M.S.F. and E. Yuzbashyan, PRL (2012)
II. M.S.F., H.-Y. X., and Y.-Z. C., PRB (2014)
III. Y.-Z. C. and M.S.F., PRB (2014)
Topological phases in materials physics:

- **Bulk electronic structure is insulating, “knotted”**
- **Knot unties at the surface:** metallic (gapless) surface states with **anomalous properties**
- **Surface or edge properties tied to anomalous symmetry:** “Holographic” encoding of bulk topology
K. von Klitzing 1980, Noble Prize in Physics 1985

• 2D electron gas, large magnetic field
• Anomalous “chiral” edge state
• Topologically quantized Hall resistance

$$R_H = \frac{h}{\nu e^2}, \quad \nu \in 1, 2, \ldots$$
How to understand?

A. Topology (TKNN invariant, Chern number, etc)

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B. **Landau levels….?**

- Electrons $E \times B$ drift
- $N_\phi$ degenerate states per LL
- Fermi energy pinned to LL

\[
R_H = \frac{h}{\nu e^2}, \quad \nu \in 1, 2, \ldots
\]
Integer Quantum Hall Effect

How to understand?

A. **Topology (TKNN invariant, Chern number, etc)**

B. **Landau levels….?**

\[ R_H = \frac{\hbar}{\nu e^2}, \quad \nu \in 1, 2, \ldots \]

- Electrons \( E \times B \) drift
- \( N_\phi \) degenerate states per LL
- Fermi energy pinned to LL

\[ \therefore R_H = \frac{B}{n e c} \]
Anderson Localization: Basics

**Weak disorder:**

*Extended states*

\[ |\psi|^2(r) \]

\[ V(r) \]
Philip Anderson, 1958:

A sufficiently strong random potential $V(r)$ exponentially localizes all single particle states (at a given energy)
Extended states at $\varepsilon_F$: (Dirty) Metal!

Weak disorder: Extended states

Philip Anderson, 1958:
A sufficiently strong random potential $V(r)$ exponentially localizes all single particle states (at a given energy)

Strong disorder or low dimensions: Localized states
Charaterized by localization length $\xi_{loc}$

Localized states at $\varepsilon_F$: Anderson Insulator!
Disorder and localization are necessary to observe the quantum Hall effect!

- States away from LL center are Anderson localized
- Delocalized bulk states only at the interplateau (topological phase) transition
- What carries the quantized Hall current in a plateau?
Chiral edge state: Half of a normal quantum wire

Normal 1D quantum wire (e.g., carbon nanotube)

- Left, right moving electrons
- Scattering (e-e, e-impurity) can change left-mover into right-mover, vice-versa
- **Left and right movers are not separately conserved**

Total charge (left + right) is conserved, but no guarantee it will flow

(Anderson localization)
Chiral edge states in the quantum Hall effect

- Left, right moving electrons separated by macroscopic bulk
- Scattering ineffective: left mover cannot be scattered into right
- Left and right movers separately conserved
  [anomalous U(1) symmetry]

No scattering:
Left, right edges are perfect quantum wires

\[ \sigma_{xy} = \frac{e^2}{h} \]
Bulk wavefunctions at the integer quantum Hall plateau transition

- States away from LL center are Anderson localized ("Topological Anderson Insulator")
- **Delocalized states** at the transition are extended (non-zero regular conductance), but highly rarified and inhomogeneous

Prange 1987
Bulk wavefunctions at the integer quantum Hall plateau transition

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\[ |\psi(\mathbf{r})|^2 \]

for an extended state (numerics)
Wavefunction multifractality: integer quantum Hall plateau transition

Statistics of rare peaks and valleys can be encoded the “singularity spectrum” $f(\alpha)$

\[ |\psi(\mathbf{r})|^2 \]

for an extended state (numerics)

**Interpretation:** over a fractal level set of measure $\mathcal{L} f(\alpha)$,

\[ (0 \leq f(\alpha) \leq 2) \]

the wavefunction probability scales as

\[ |\psi|^2 = c L^{-\alpha} \]
Wavefunction multifractality: integer quantum Hall plateau transition

Statistics of rare peaks and valleys can be encoded the “singularity spectrum” $f(\alpha)$

Self-averaging and universal

2D critical delocalization: Universal wavefunction multifractality

Chamon, Mudry, Wen 1996; Mirlin and Evers 2000; Obuse, Subramaniam, Furusaki, Gruzberg, Ludwig 2008; Evers, Mildenberger, Mirlin 2008
Can define a multifractal spectrum whenever there is some probability density distributed over a (possibly strange) set

- **Strange attractors**
  Wiklund and Elgin 1996; Gratrix and Elgin 2004

- **Diffusion–Limited Aggregation**
  Nittmann, Stanley, Touboul, Daccord 1987

- **Forced Rayleigh–Bénard convection**
  Glazier, Gunaratne, and Libchaber 1988

- **Financial markets**
Superconductivity

Collective motion of loosely bound electron pairs at low temperatures

- Superfluidity: Electrical resistance is zero
- No heat or spin transport in the superfluid
- Topological superconductor: Theorized to possess a charge neutral surface fluid of unpaired “Majorana” fermions
Superconductivity
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Old idea: “P + i P” topological superconductivity in 2D

1. Chiral 1D Majorana edge states quantized thermal Hall conductance
   \[ \kappa_{xy} = |\nu| \frac{\pi^2 k_B T}{12\pi \hbar} \]

2. Isolated Majorana zero modes (vortices)
   Possible experimental realizations
   - 5/2 Fractional Quantum Hall effect (??)
   - Thin film \(^3\)He-A, Sr\(_2\)RuO\(_4\) (??)

Moore and Read 91, Read and Green 00
Volovik 88, Rice and Sigrist 95
Topological Superconductor: Gapped bulk, Majorana fluid boundary

**Superconductivity**
Collective motion of loosely bound electron pairs at low temperatures

- **Superfluidity**: Electrical resistance is zero
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“New” idea: 3D Bulk topological superconductivity

Integer-valued winding number \( \nu \in \mathbb{Z} \)

2D Majorana surface fluid, envelopes bulk

- Transport properties?
- Stability? (Exposed crystal surface: disorder)

**Experimental realizations**

- Helium 3B (neutral topological superfluid)
- \( \text{Cu}_x\text{Bi}_2\text{Se}_3 \) (?) (doped top. insulator)

Schnyder, Ryu, Furusaki, Ludwig 2008; Kitaev 2009
Volovik 1988
Fu and Berg 2010

Mai-Linh Doan, Wikipedia
For a 3D topological superconductor with bulk winding number $\nu$, what do the Majorana surface states “look like?”
A lot like graphene!

- Unpaired surface Majorana fermion quasiparticles
- $|\nu| = 2k$ “colors,” $k = (1,2,3,...)$ (class CI)

Low energy surface Andreev state Hamiltonian:

$$H = \int d^2r \, \Psi^\dagger \left( -i\hat{\sigma}^1 \partial_x - i\hat{\sigma}^2 \partial_y \right) \Psi \equiv \Psi^\dagger \hat{h} \Psi$$

Majorana fermion carries Pseudospin ($\sigma$) and Color ($\kappa$) indices

$$\Psi_\kappa = \begin{bmatrix} C_{\uparrow,\kappa} \\ C_{\downarrow,\kappa} \end{bmatrix}$$

$$1 \leq \kappa \leq |\nu|$$

“Anomalous” chiral symmetry (= physical time-reversal):

$$-\hat{\sigma}^3 \hat{h} \hat{\sigma}^3 = \hat{h}$$

Schnyder, Ryu, Furusaki, Ludwig 08
Bernard, LeClair 02
**Effects of disorder**

- Junk is unavoidable at the surface!
- *Any* non-magnetic (time-reversal preserving) surface perturbation: intercolor vector potential $\hat{t}_\kappa^i A_i(r)$!

\[
H = \int d^2r \bar{\Psi} \left( -i\sigma \cdot \nabla + A_i \cdot \sigma \hat{t}_\kappa^i \right) \Psi = H_0 + \int d^2r \left( J_\kappa^i \bar{A}_i + \bar{J}_\kappa^i A_i \right)
\]

Sources of $\hat{t}_\kappa^i A_i(r)$:

- Impurities, vacancies
- External electric fields
- Edge, corner, dislocation potentials

“Quenched” 2+1-D QCD: Dirac fermions in a sea of frozen gauge fluctuations
**Surface Majorana fluid can carry spin or heat**

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. Multiple scattering off of impurities (weak localization)
Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. Multiple scattering off of impurities (weak localization)

Kubo formula for dc spin conductivity:

\[
\sigma = \frac{-1}{4\pi L^d} \int_{r_1,r_2} \text{Re} \left\{ \text{Tr} \left[ \hat{\sigma}^\alpha \hat{G}^{(A)}(0; r_1, r_2) \hat{\sigma}^\alpha \hat{G}^{(R)}(0; r_2, r_1) - \hat{\sigma}^\alpha \hat{G}^{(R)}(0; r_1, r_2) \hat{\sigma}^\alpha \hat{G}^{(R)}(0; r_2, r_1) \right] \right\}
\]

Components:
Retarded, advanced Green’s functions

\[
\hat{G}^{(R,A)}(\varepsilon; r_1, r_2) \equiv \langle r_1 | \frac{1}{\varepsilon \pm i\eta - \hat{h}} | r_2 \rangle
\]

\[
\hat{h} = -i\hat{\sigma} \cdot \nabla + A_i(r) \cdot \sigma \hat{t}_i
\]
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\]

Anomalous form of time-reversal symmetry

Retarded, advanced interchangable:

\[
-\hat{\sigma}^3 \hat{G}^{(A)}(\varepsilon; r_1, r_2) \hat{\sigma}^3 = \hat{G}^{(R)}(-\varepsilon; r_2, r_1)
\]

\[
\sigma = -\frac{1}{\pi} \lim_{r\to r'} \text{Im} \left\{ \text{Tr} \left[ (r - r') \cdot \hat{\sigma} \hat{G}^{(R)}(0; r, r') \right] \right\} = \frac{|\nu|}{2\pi^2} 2+0-D \text{ Chiral anomaly}
\]

Universal spin, thermal conductivity, neglecting interactions

\[
\sigma_s = \frac{|\nu|}{\pi \hbar} \left( \frac{\hbar}{2} \right)^2, \quad \kappa = \frac{|\nu|}{\pi \hbar} \frac{\pi^2 k_B^2 T}{3}
\]

Tsvelik (1995)
Ostrovsky, Gornyi, Mirlin (2006)
Foster, Xie, and Chou (2014)
Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. Multiple scattering off of impurities (weak localization)
2. Scattering off of impurity-induced density Friedel oscillations (Altshuler-Aronov corrections)

Altshuler and Aronov 1985
Aleiner, Altshuler, and Gershenson 1999
Zala, Narozhny, and Aleiner 2001
Surface Majorana fluid can carry spin or heat

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1. Multiple scattering off of impurities (weak localization)
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Anomalous time-reversal symmetry:

$$-\hat{\sigma}^3 \hat{G}^{(A)}(\varepsilon; r_1, r_2) \hat{\sigma}^3 = \hat{G}^{(R)}(-\varepsilon; r_2, r_1)$$

Xie, Chou, and Foster (2015)
Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. Multiple scattering off of impurities (weak localization)
2. Scattering off of impurity-induced density Friedel oscillations (Altshuler-Aronov corrections)

Anomalous time-reversal symmetry: No Majorana “density” (mass, spin, color) can ripple (or become non-zero)!

Universal spin, thermal conductivities:

$$\sigma_s = \frac{|\nu|}{\pi \hbar} \left( \frac{\hbar}{2} \right)^2$$

$$\kappa = \frac{|\nu|}{\pi \hbar} \frac{\pi^2 k_B^2 T}{3}$$

Xie, Chou, and Foster (2015)
Universal transport: Disorder has no effect?

Ordinary 2D electron gas with time reversal symmetry (no magnetic field): arbitrarily weak disorder localizes all wavefunctions.
Universal transport: Disorder has no effect?

Surface Majorana states cannot be localized (topological protection). Instead: critical delocalization

Chou and Foster (2014)
2D Conformal field theory solution via \textit{conformal embeddings}

| Class | Embedding | \(|\nu|\) |
|-------|-----------|---------|
| CI    | $SO(4nk)_1 \supset Sp(2n)_k \oplus Sp(2k)_n$ | \(2k\) \((k \geq 1)\) |
| AI\,III | $U(nk)_1 \supset U(n)_k \oplus SU(k)_n$ | \(k\) \((k \geq 2)\) |
| DI\,III | $SO(nk)_1 \supset SO(n)_k \oplus SO(k)_n$ | \(k\) \((k \geq 3)\) |

Winding number \(|\nu| = \# \text{ colors}\)

- “Fractionalization”: Color sector “localizes”

Wave functions are multifractal; exact spectra computed via CFT:

**Exact Results**

\[
f(\alpha) = 2 - \frac{(\alpha - 2 - \theta_k)^2}{4\theta_k}
\]

| Class | \(|\nu|\) | \(\theta_k\) |
|-------|---------|-----------|
| CI    | \(2k\) | $\frac{1}{2(k + 1)}$ |
| AI\,III | \(k \geq 1\) \(\frac{k - 1}{k^2} + \lambda_A\) |
| DI\,III | \(k \geq 3\) | $\frac{1}{k - 2}$ |

Reviewed in (e.g.)

J. Fuchs, \textit{Affine Lie Algebras and Quantum Groups}

Nersesyan, Tsvelik, Wenger 94

Foster, Yuzbashyan 12
Mudry, Chamon, Wen 96
Caux, Kogan, Tsvelik 96
Foster, Xie, Chou 14
Minimal case: 2 valley Dirac (Classes CI and AIII)

CFT predictions:
- Global density of states

\[ \nu(\varepsilon) \sim |\varepsilon|^{\eta}, \quad \eta = \frac{1 - 4\lambda_A}{7 + 4\lambda_A} \]

- Multifractal spectrum

\[ f(\alpha) = \frac{8(\alpha^+ - \alpha)(\alpha - \alpha^-)}{(\alpha^+ - \alpha^-)^2}, \quad \alpha^\pm = (\sqrt{2} \pm \sqrt{\theta_2})^2 \]

Numerical scheme:
Momentum-space disordered Dirac fermion (avoids fermion doubling)

Numerical tests: Critical DOS, multifractal scaling

Bardarson, Tworzydlo, Brouwer, Beenakker 07
Nomura, Koshino, Ryu 07

Chou and Foster 2014
Extended, multifractal surface states: 
No Anderson localization = topological protection! ...BUT

Add generic, weak interparticle interactions, consistent with bulk symmetries [time-reversal, spin SU(2) for CI]

\[ H_I = U \int d^2r \Psi^\dagger_\alpha \Psi_\beta \Psi^\dagger_\gamma \Psi_\delta \]
**Physical picture: Chalker scaling, multifractality, and interactions**

- **Chalker scaling:** Overlapping peaks and valleys in multifractal eigenstates with nearby energies

\[
\lim_{L \to \infty} \int d^2r \left| \psi_0(r) \right|^2 \left| \psi_\varepsilon(r) \right|^2 \sim \frac{\varepsilon^{-\mu}}{L^2}, \quad \mu = \frac{2 - \tau(2)}{2}
\]

Probability peaks in **different** wavefunctions tend to cluster

Chalker, Daniell 88
Chalker 90
Cuevas, Kravtsov 07
Feigelman, Ioffe, Kravtsov, Yuzbashyan 07
Feigelman, Ioffe, Kravtsov, Cuevas 10

Chou and Foster (2014)
• **Chalker scaling:** Overlapping peaks and valleys in multifractal eigenstates with nearby energies

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\]

Probability peaks in *different* wavefunctions tend to cluster

• Why scaling theory of localization works

• Enhances interaction matrix elements –**instabilities**!

• *Anderson insulator*: No overlap for nearby energies

\[
\left| \psi_{\varepsilon}(\mathbf{r}) \right|^2 \left| \psi_{\varepsilon'}(\mathbf{r}) \right|^2 \sim 0, \quad 0 < |\varepsilon - \varepsilon'| \ll \delta_l
\]
Weak disorder and interactions can sabotage topological protection!

- **Result:** Not always protected. Even weak disorder and weak interactions can destroy some surface states.

Class CI Topological superconductors: Majorana surface fluid always unstable for any disorder, interactions, winding number.

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Foster, Yuzbashyan (2012)
Weak disorder and interactions can sabotage topological protection!

- **Result:** Not always protected. Even weak disorder and weak interactions can destroy some surface states.

**Class CI Topological superconductors:** Majorana surface fluid always unstable for any disorder, interactions, winding number.

**Classes AIII, DIII:** Stable surface states.
2D Majorana liquid theory

- Surface states of a bulk topological superconductor
- Universal transport coefficients encode bulk winding number
- Combined effects of disorder and interactions can lead to instabilities
2D Majorana liquid theory

- Surface states of a bulk topological superconductor
- Universal transport coefficients encode bulk winding number
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3D Topological superconductivity: close analog of the integer quantum Hall effect

- Is there a fractional analog? (Bulk with topological order; gapless surface fluid with fractionalized transport coefficients)
- What about gapless (nodal) “topological” superconductor surface states?

- Materials beyond Cu$_x$Bi$_2$Se$_3$ ?!