RICE

THE Welch FOUNDATION

Alfred P. Sloan FOUNDATION

A 3D integer quantum Hall effect: Universal spin and heat transport at the surface of a topological superconductor

or:

2D Majorana liquid theory

Matthew S. Foster

Rice University

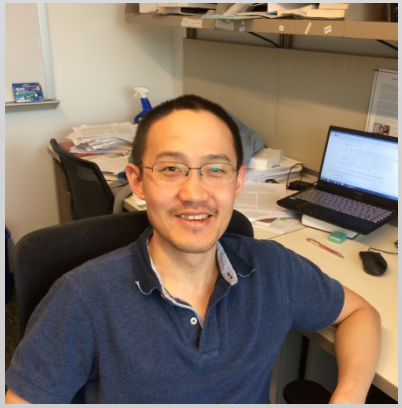
February 3rd, 2015

A 3D integer quantum Hall effect: Universal spin and heat transport at the surface of a topological superconductor

or:

2D Majorana liquid theory

Hong-Yi Xie, Yang-Zhi Chou, and Matthew S. Foster



Rice University

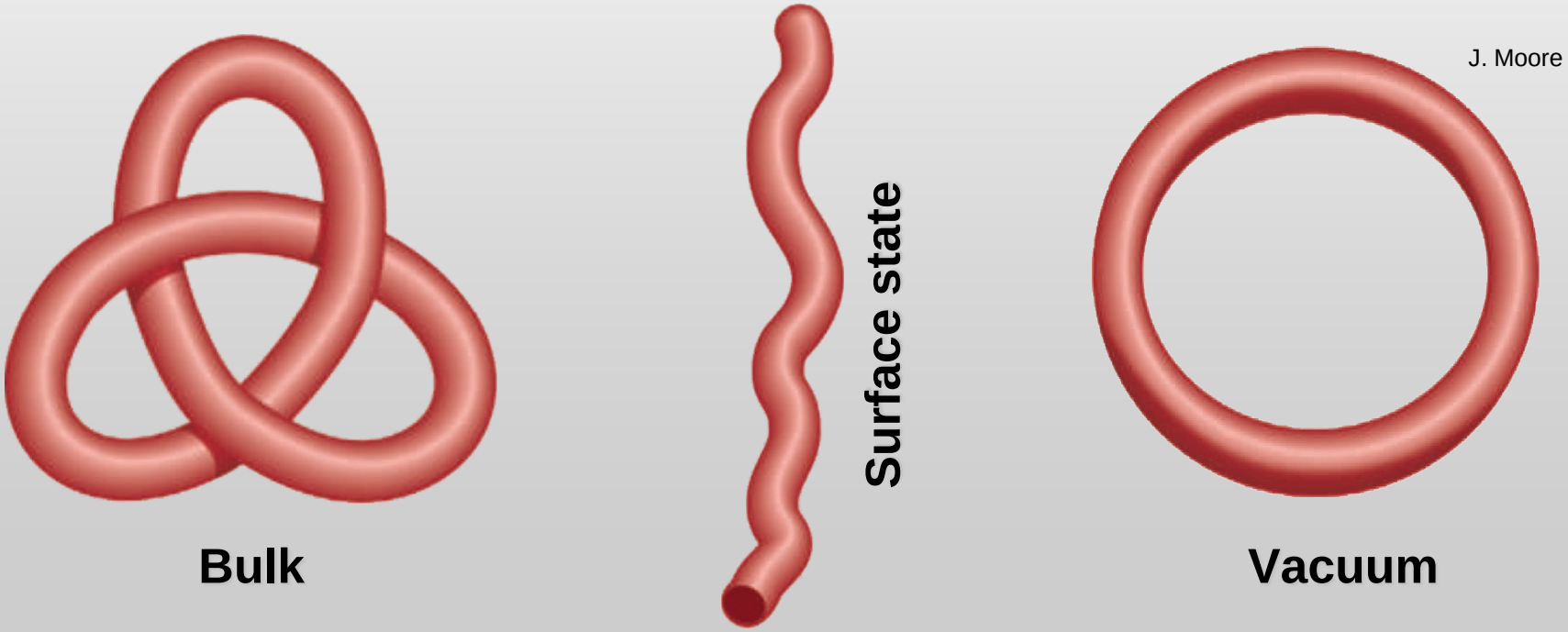
February 3rd, 2015

I. M.S.F. and E. Yuzbashyan, PRL (2012)
II. M.S.F., H.-Y. X., and Y.-Z. C., PRB (2014)
III. Y.-Z. C. and M.S.F., PRB (2014)
IV. H.-Y. X, Y.-Z. C., and M.S.F., PRB (2015)

Quantum Topology in Materials

Topological phases in materials physics:

- Bulk electronic structure is insulating, “knotted”
- Knot unties at the surface:
  metallic (gapless) surface states with anomalous properties
- Surface or edge properties tied to anomalous symmetry:
  “Holographic” encoding of bulk topology

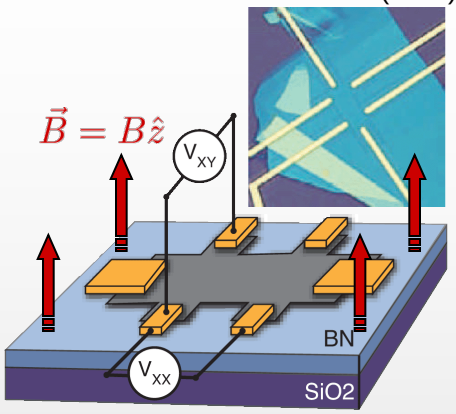


Integer Quantum Hall Effect

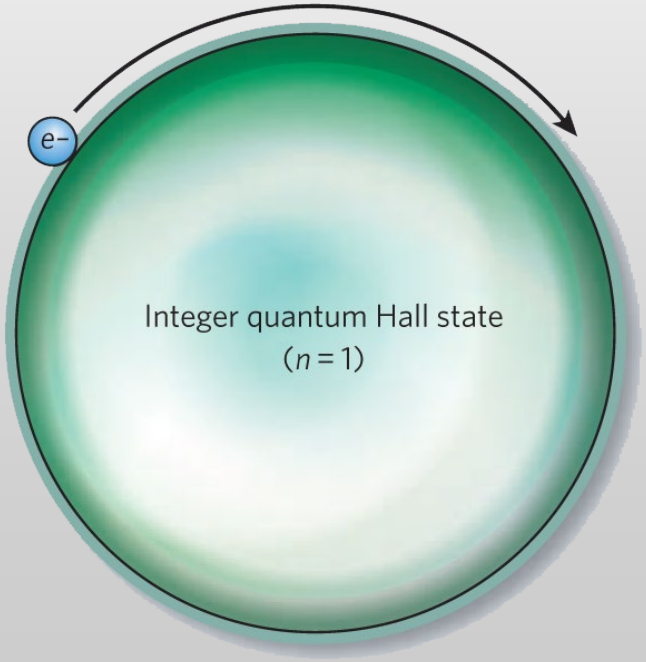
K. von Klitzing 1980, Noble Prize in Physics 1985

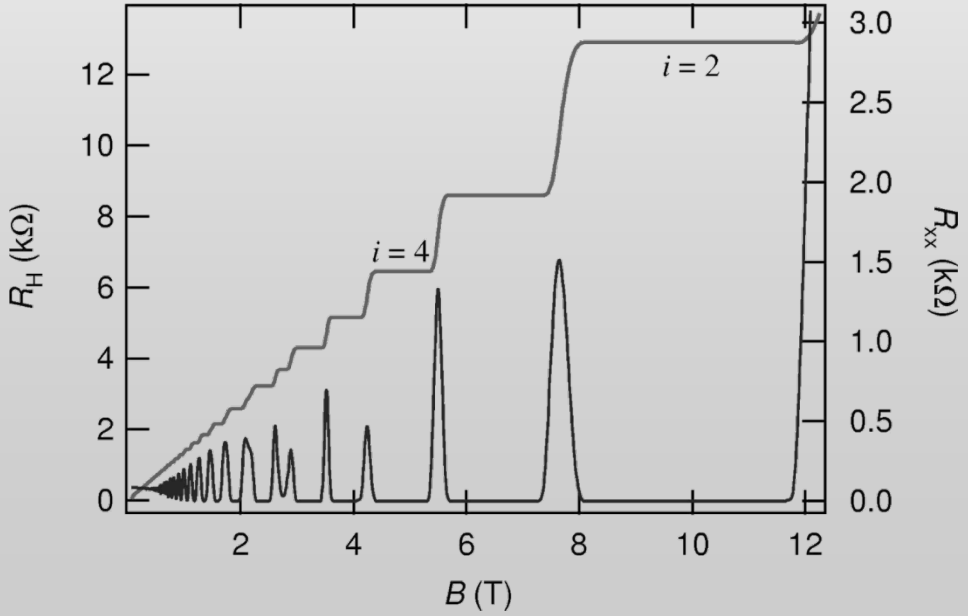
- **2D electron gas, large magnetic field**
- **Anomalous “chiral” edge state**
- **Topologically quantized Hall resistance**

Dean et al. (2010)



$$R_H = \frac{h}{\nu e^2}, \quad \nu \in 1, 2, \ldots$$

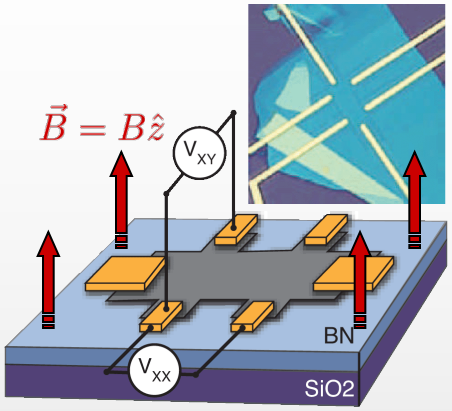




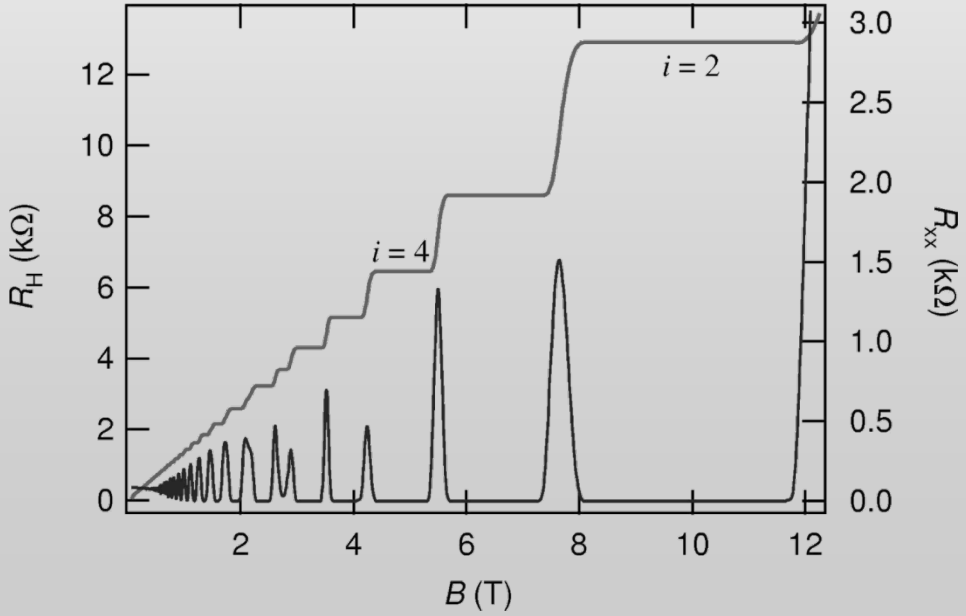
Integer Quantum Hall Effect

How to understand?

A. Topology (TKNN invariant, Chern number, etc)



$$R_H = \frac{h}{\nu e^2}, \quad \nu \in 1, 2, \ldots$$

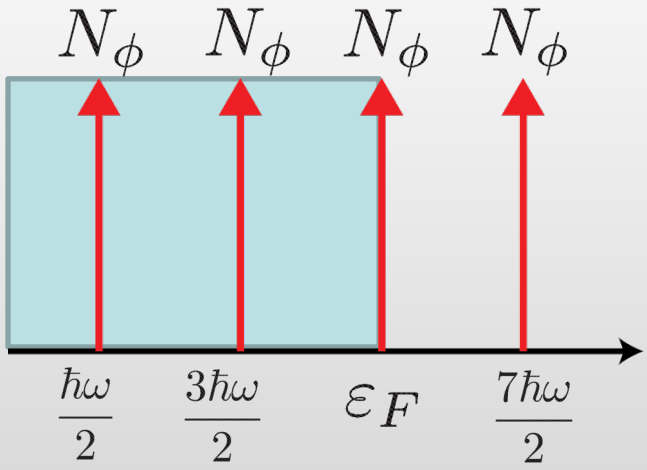


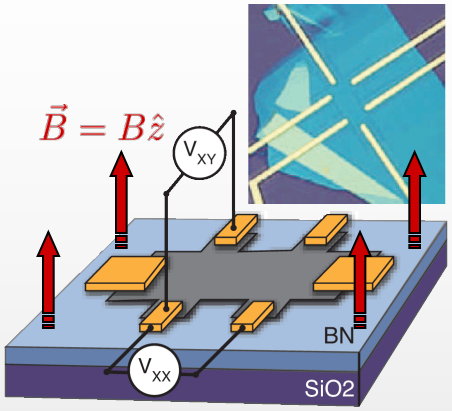
Integer Quantum Hall Effect

How to understand?

~~**A. Topology (TKNN invariant, Chern number, etc)**~~

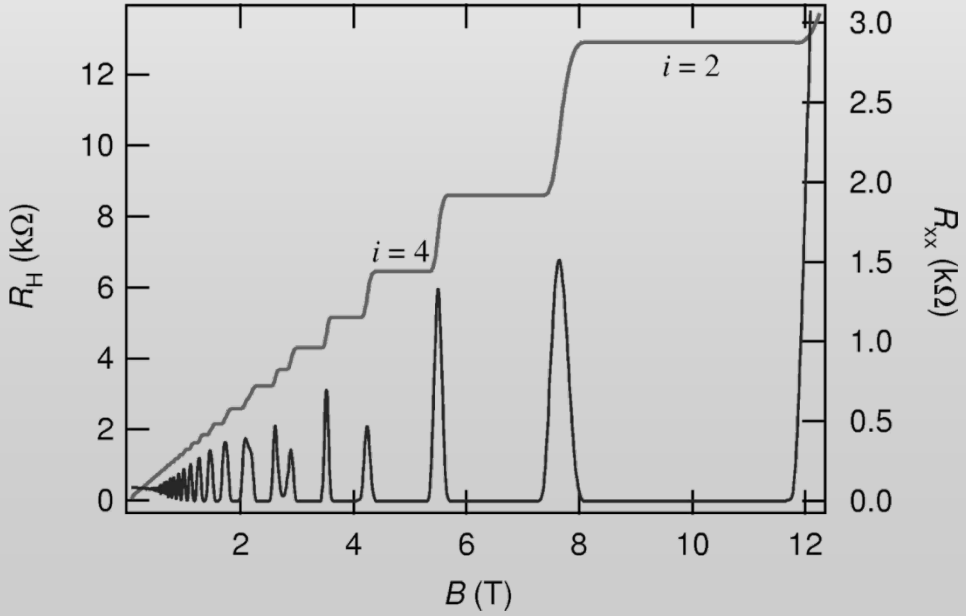
B. Landau levels….?





$$R_H = \frac{h}{\nu e^2}, \quad \nu \in 1, 2, \ldots$$

- **Electrons *E* x *B* drift**
- N_ϕ **degenerate states per LL**
- **Fermi energy pinned to LL**

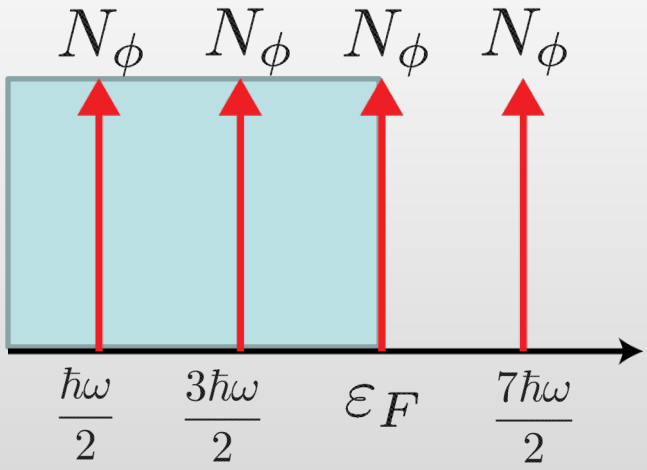


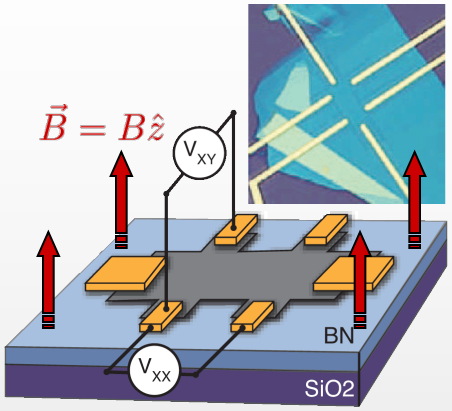
Integer Quantum Hall Effect

How to understand?

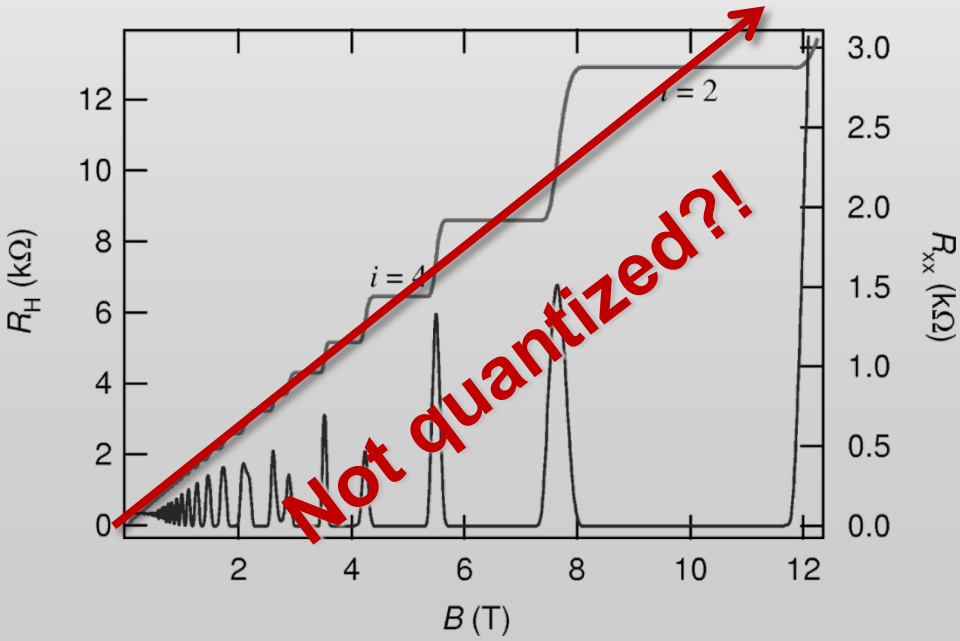
~~**A. Topology (TKNN invariant, Chern number, etc)**~~

B. Landau levels….?





$$R_H = \frac{h}{\nu e^2}, \quad \nu \in 1, 2, \ldots$$

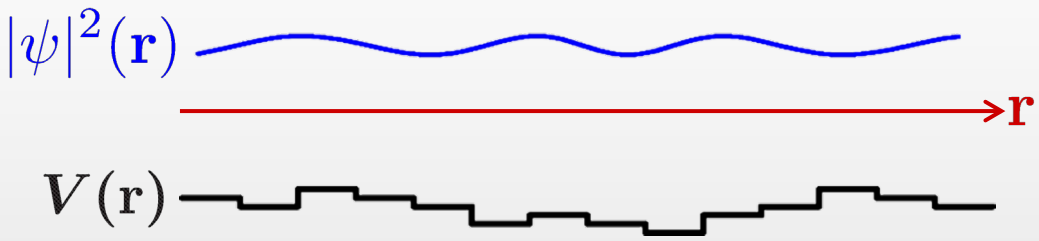


- **Electrons *E* x *B* drift**
- N_ϕ **degenerate states per LL**
- **Fermi energy pinned to LL**

$$\therefore \; R_H = \frac{B}{nec}$$

Weak disorder:

Extended states



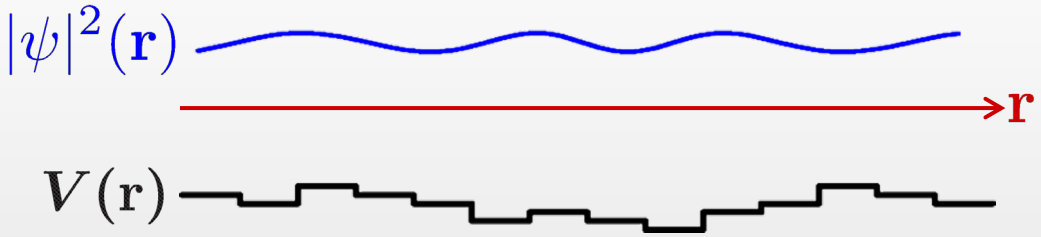
Anderson Localization: Basics

Philip Anderson, 1958:

A sufficiently strong random potential $V(\mathrm{r})$ exponentially localizes all single particle states (at a given energy)

Weak disorder:

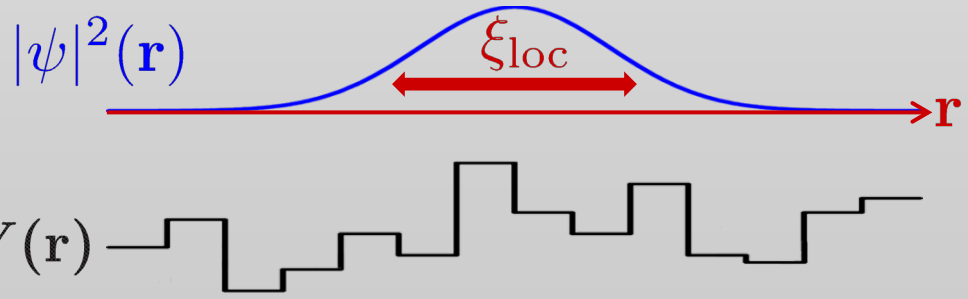
Extended states



Strong disorder or low dimensions:

Localized states

Characterized by localization length ξ_{loc}



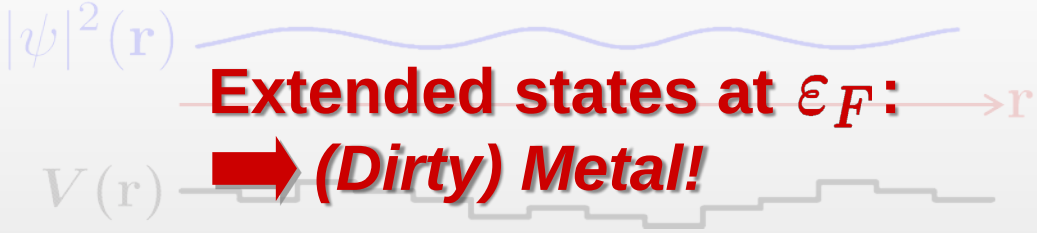
Anderson Localization: Basics

Philip Anderson, 1958:

A sufficiently strong random potential $V(\mathrm{r})$ exponentially localizes all single particle states (at a given energy)

Weak disorder:

Extended states



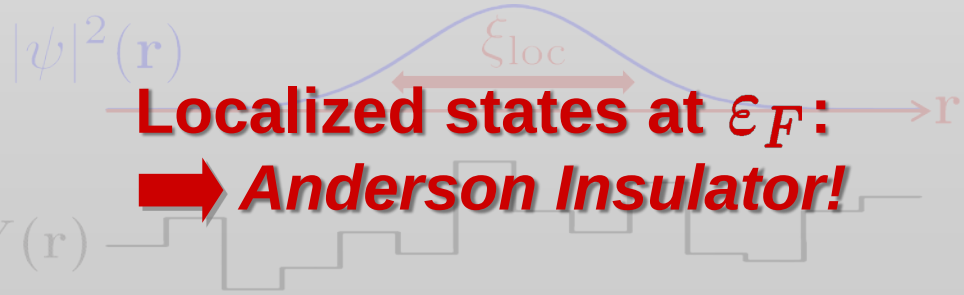
Extended states at ε_F:

➡ ***(Dirty) Metal!***

Strong disorder or low dimensions:

Localized states

Characterized by localization length ξ_{loc}



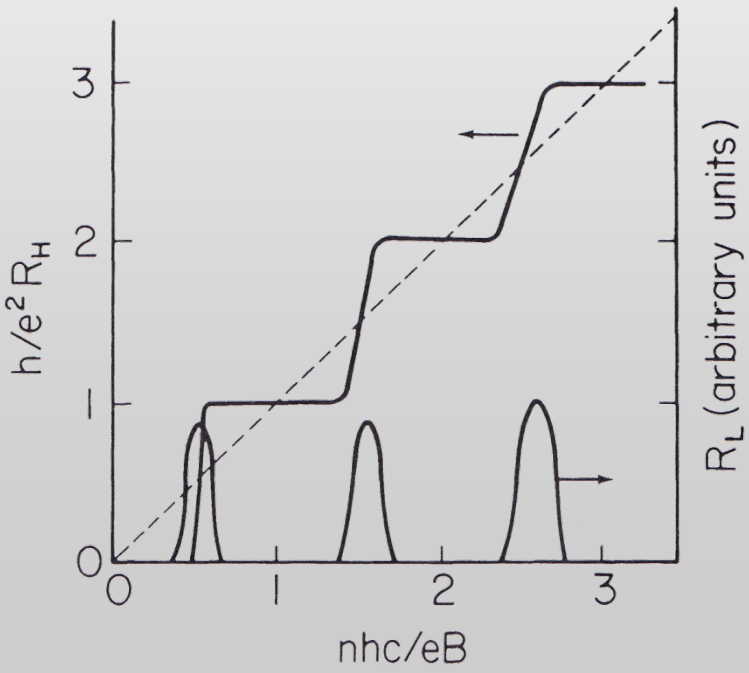
Localized states at ε_F:

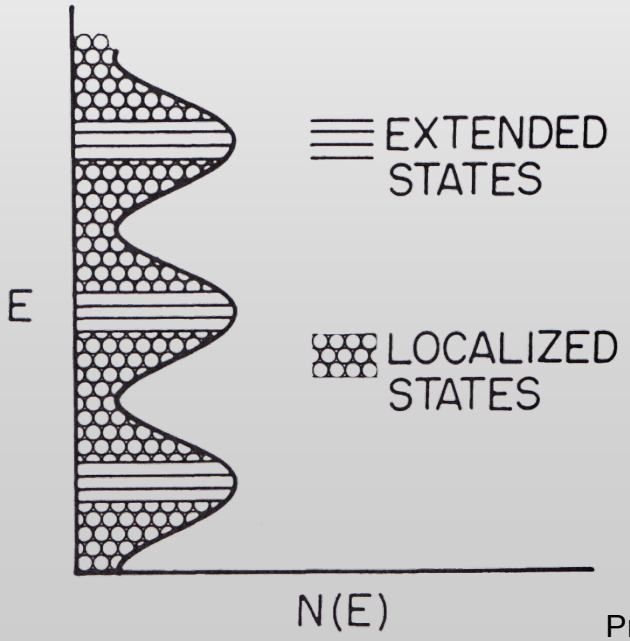
➡ ***Anderson Insulator!***

Integer quantum Hall effect: *"Topological Anderson insulator"*

Disorder and localization are necessary to observe the quantum Hall effect!

- **States away from LL center are Anderson localized**
- ***Delocalized bulk states* only at the interplateau (topological phase) transition**
- **What carries the quantized Hall current in a plateau?**



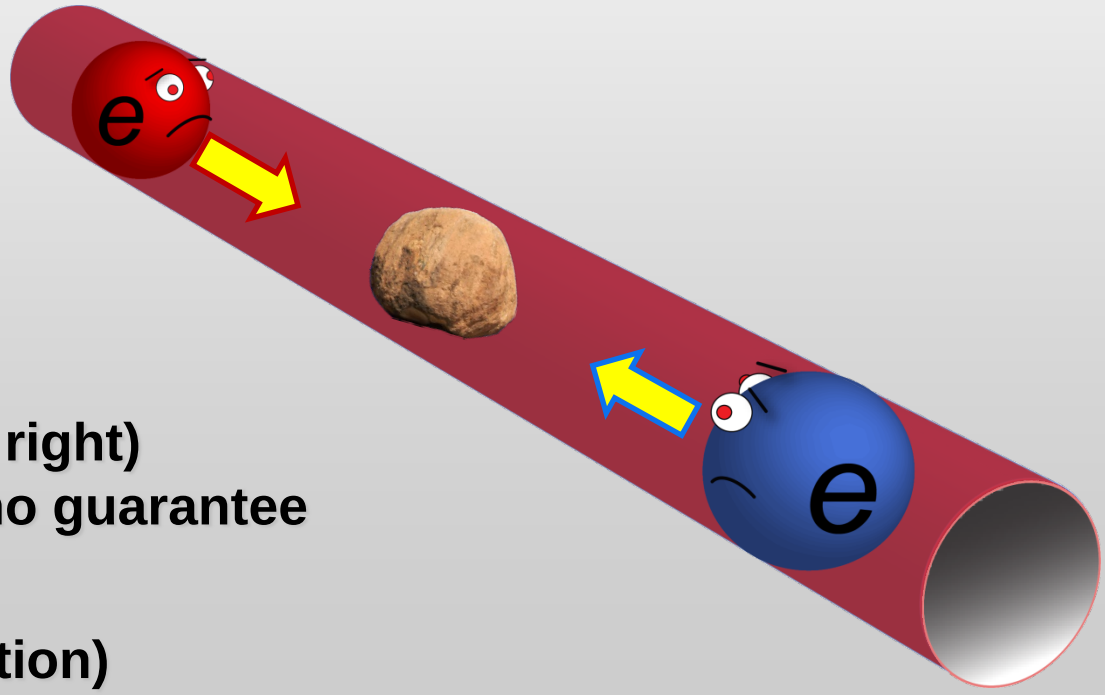


Prange 1987

Chiral edge state: Half of a normal quantum wire

Normal 1D quantum wire (e.g., carbon nanotube)

- **Left, right moving electrons**
- **Scattering (e-e, e-impurity) can change left-mover into right-mover, vice-versa**
- **Left and right movers are not separately conserved**



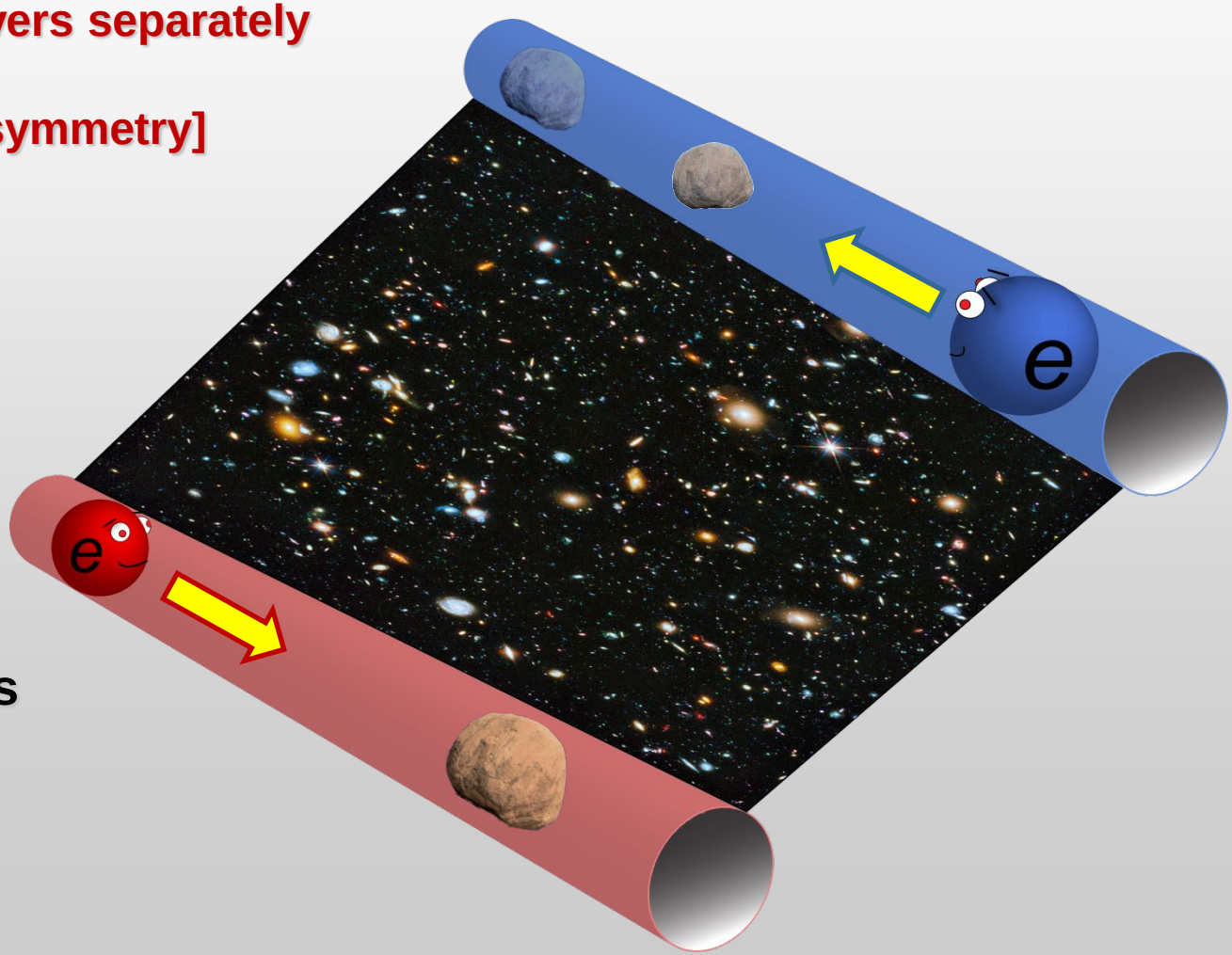
Total charge (left + right) is conserved, but no guarantee it will flow

(Anderson localization)

Chiral edge state: Half of a normal quantum wire

Chiral edge states in the quantum Hall effect

- Left, right moving electrons separated by macroscopic bulk
- Scattering ineffective: left mover cannot be scattered into right
- Left and right movers separately conserved
  [anomalous U(1) symmetry]



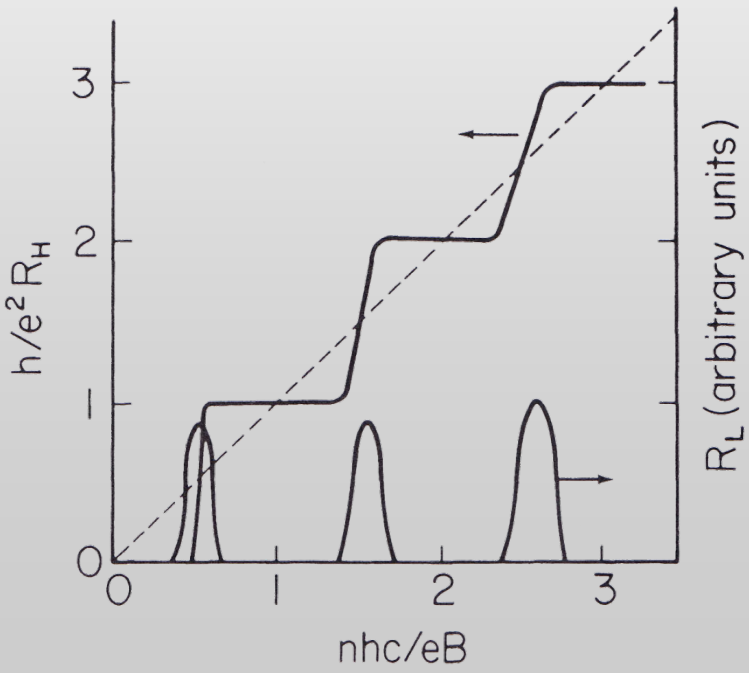
No scattering:
Left, right edges are perfect quantum wires

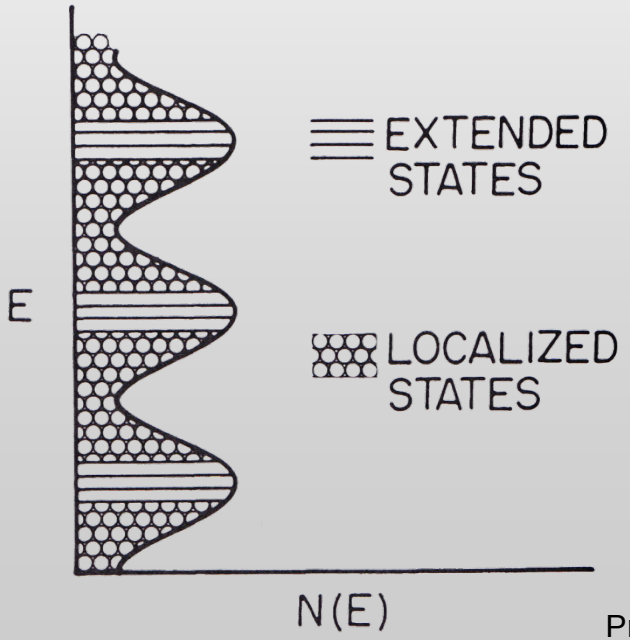
$$\sigma_{xy} = \frac{e^2}{h}$$

2D critical delocalization: Universal wavefunction multifractality

Bulk wavefunctions at the integer quantum Hall plateau transition

- **States away from LL center are Anderson localized ("Topological Anderson Insulator")**
- ***Delocalized states* at the transition are extended (non-zero regular conductance), but highly rarified and inhomogeneous**



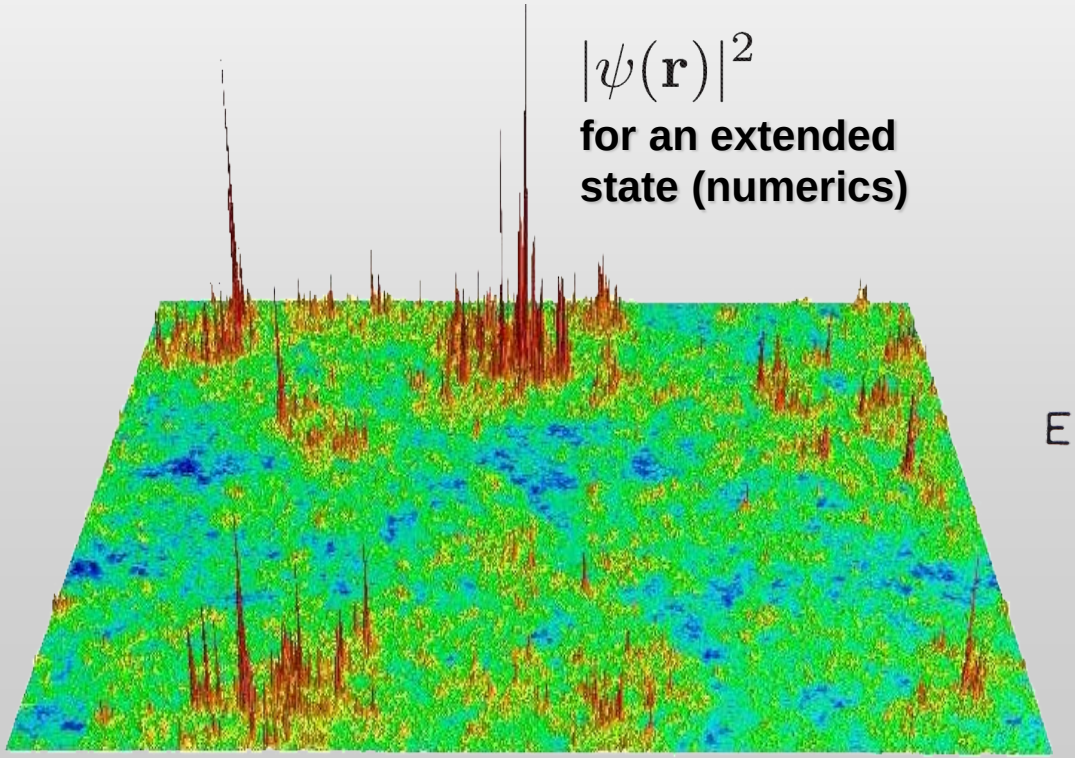


Prange 1987

2D critical delocalization: Universal wavefunction multifractality

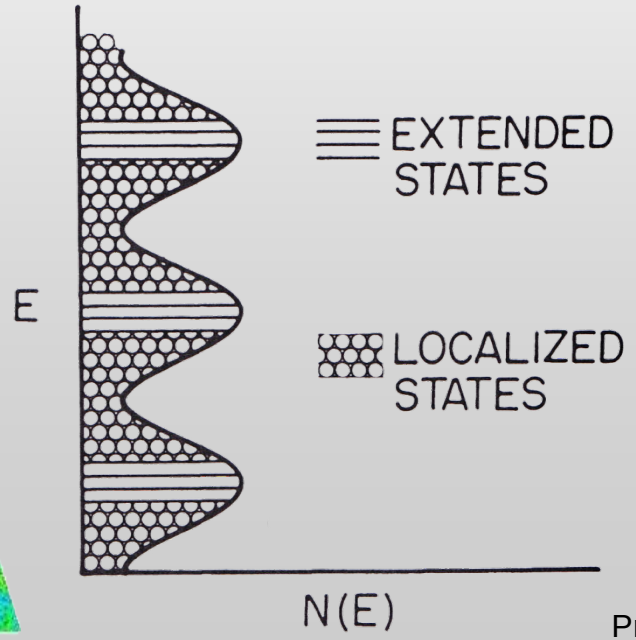
Bulk wavefunctions at the integer quantum Hall plateau transition

- **States away from LL center are Anderson localized ("Topological Anderson Insulator")**
- ***Delocalized states* at the transition are extended (non-zero regular conductance), but highly rarified and inhomogeneous**



$|\psi(\mathbf{r})|^2$ for an extended state (numerics)

F. Evers

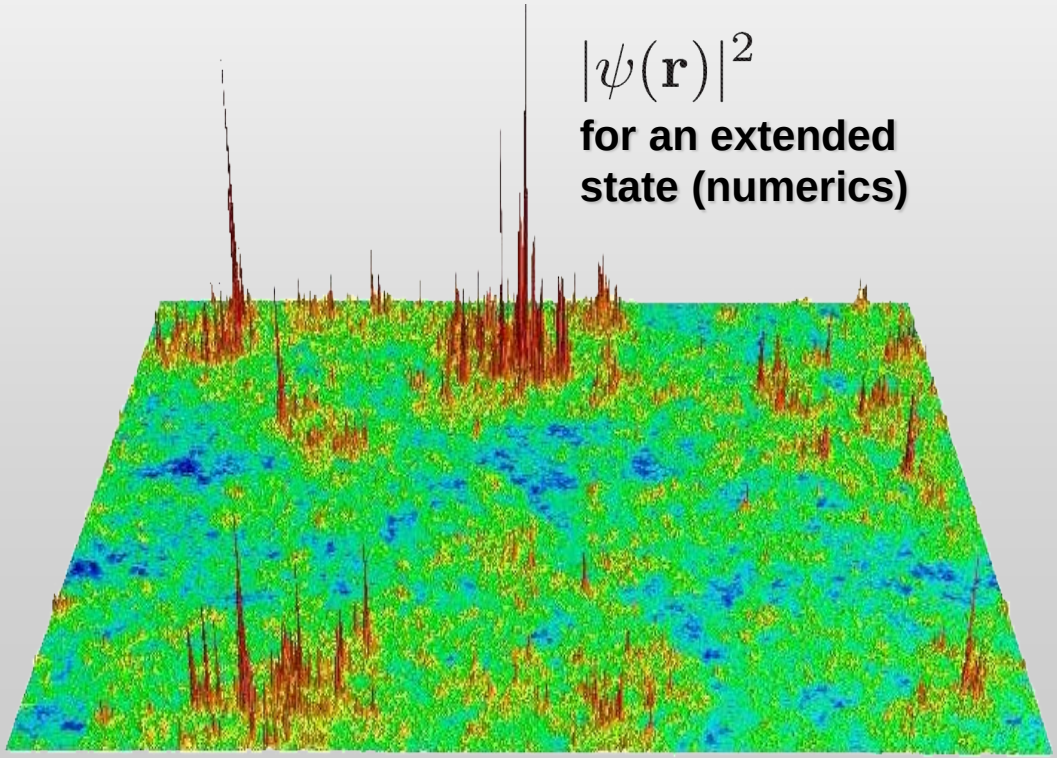


Prange 1987

2D critical delocalization: Universal wavefunction multifractality

Wavefunction multifractality: integer quantum Hall plateau transition

Statistics of rare peaks and valleys can be encoded the "singularity spectrum" $f(\alpha)$



$|\psi(\mathbf{r})|^2$ **for an extended state (numerics)**

F. Evers

***Interpretation*: over a *fractal level set* of measure** $L^{f(\alpha)}$,

$$(0 \leq f(\alpha) \leq 2)$$

the wavefunction probability scales as

$$|\psi|^2 = c\, L^{-\alpha}$$

2D critical delocalization: Universal wavefunction multifractality

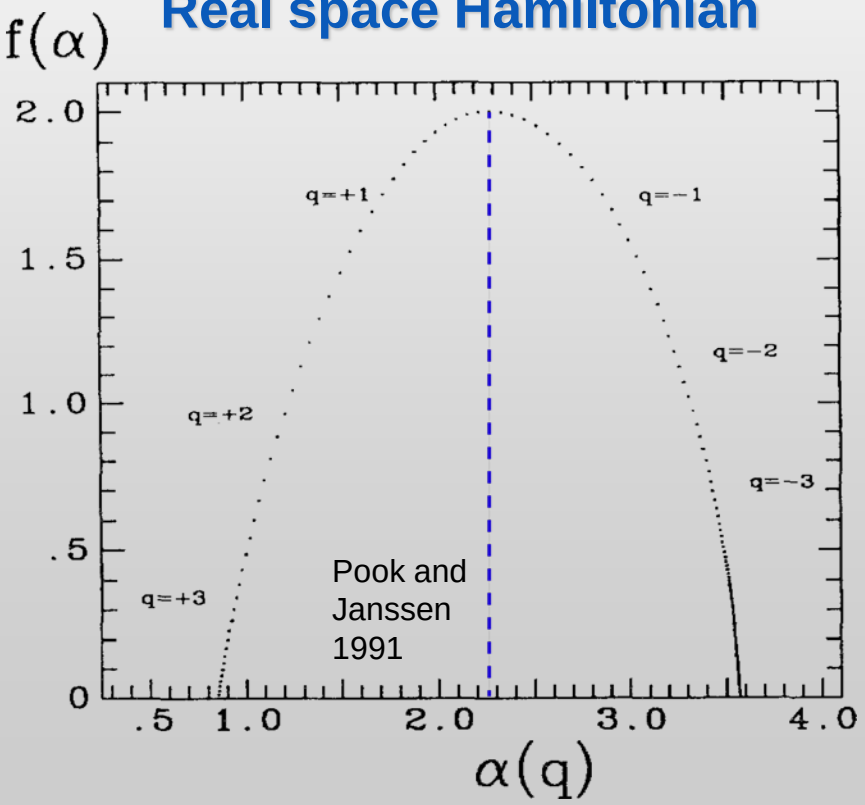
Wavefunction multifractality: integer quantum Hall plateau transition

Statistics of rare peaks and valleys can be encoded the "singularity spectrum" $f(\alpha)$

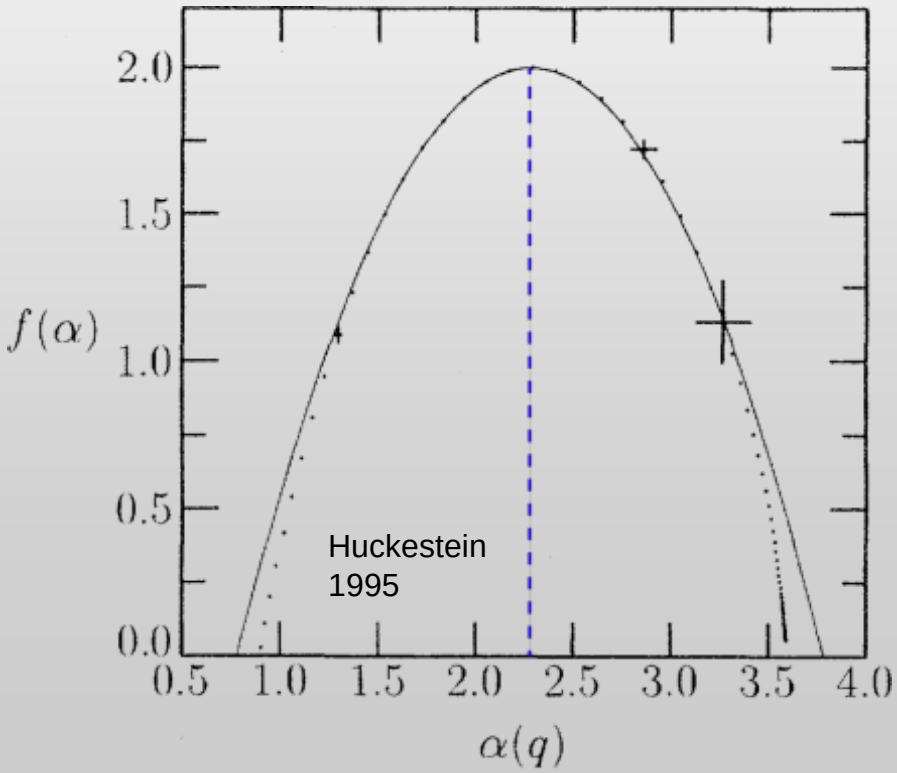
Self-averaging and universal

Chamon, Mudry, Wen 1996;
Mirlin and Evers 2000;
Obuse, Subramaniam, Furusaki, Gruzberg, Ludwig 2008;
Evers, Mildenberger, Mirlin 2008

Real space Hamiltonian



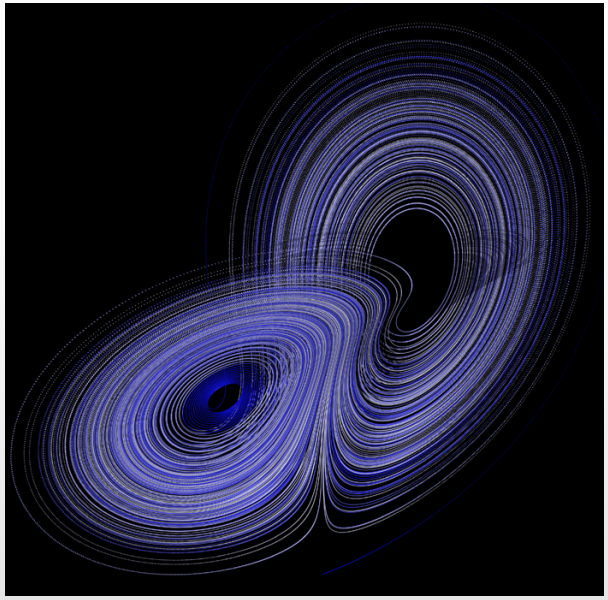
Tight-binding model



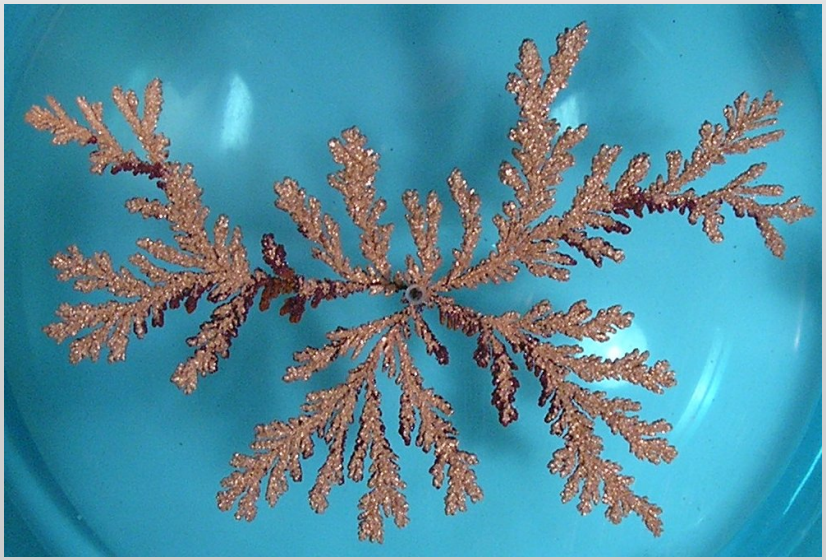
Multifractal spectra: Statistics of complex phenomena

Can define a multifractal spectrum whenever there is some probability density distributed over a (possibly strange) set

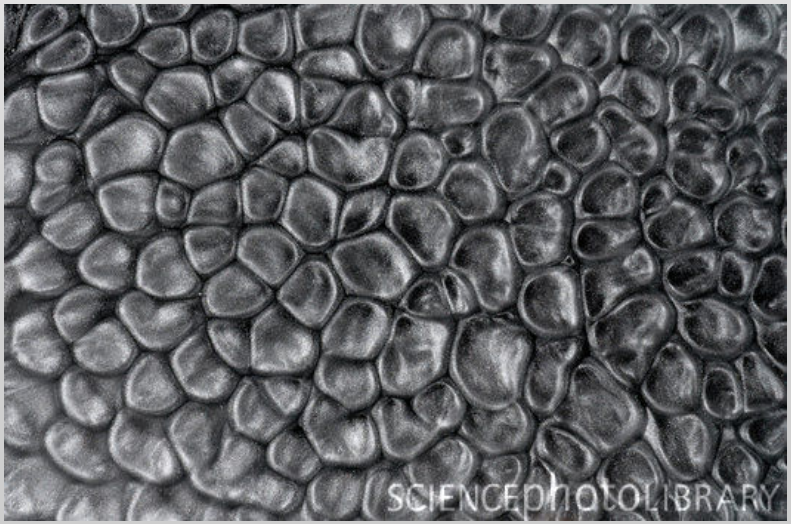
- **Strange attractors**
  Wiklund and Elgin 1996; Gratrix and Elgin 2004
- **Diffusion–Limited Aggregation**
  Nittmann, Stanley, Touboul, Daccord 1987
- **Forced Rayleigh–Bénard convection**
  Glazier, Gunaratne, and Libchaber 1988
- **Financial markets**



Marcus Fritzsch



Kevin R. Johnson 2004

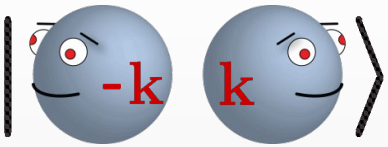


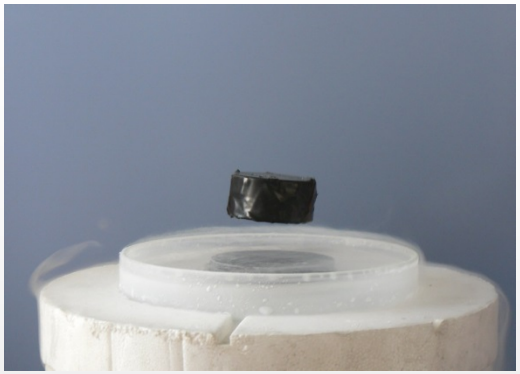
Scott Camazine

Topological Superconductor: Gapped bulk, Majorana fluid boundary

Superconductivity

Collective motion of loosely bound electron pairs at low temperatures





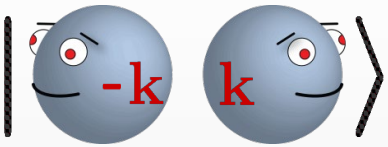
Mai-Linh Doan, Wikipedia

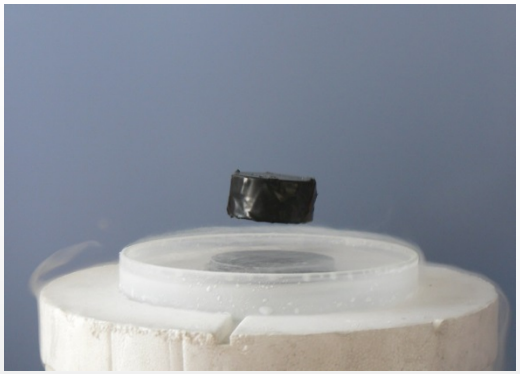
- **Superfluidity: Electrical resistance is zero**
- **No heat or spin transport in the superfluid**
- **Topological superconductor:**
  Theorized to possess a charge neutral surface fluid of unpaired “Majorana” fermions

Topological Superconductor: Gapped bulk, Majorana fluid boundary

Superconductivity

Collective motion of loosely bound electron pairs at low temperatures

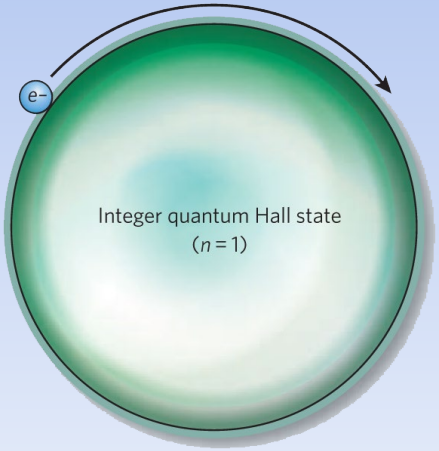




Mai-Linh Doan, Wikipedia

- **Superfluidity: Electrical resistance is zero**
- **No heat or spin transport in the superfluid**
- **Topological superconductor:**
  Theorized to possess a charge neutral surface fluid of unpaired "Majorana" fermions

Old idea: "P + i P" topological superconductivity in 2D



1. **Chiral 1D Majorana edge states quantized thermal Hall conductance** $\kappa_{xy} = |\nu| \frac{\pi^2 k_B T}{12\pi\hbar}$
2. **Isolated Majorana zero modes (vortices)**

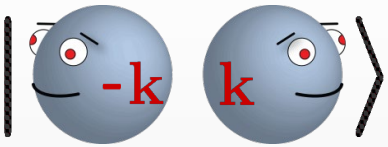
Possible experimental realizations

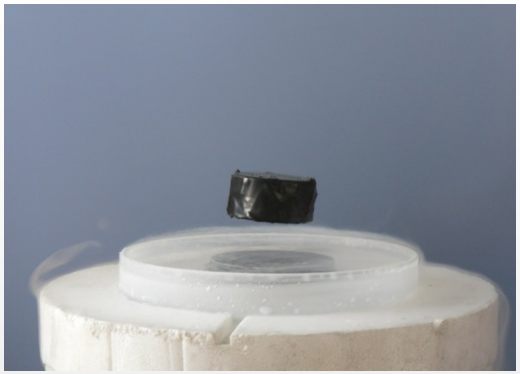
- **5/2 Fractional Quantum Hall effect (?)** — Moore and Read 91, Read and Green 00
- **Thin film ^{3}He-A, Sr_2RuO_4 (?)** — Volovik 88, Rice and Sigrist 95

Topological Superconductor: Gapped bulk, Majorana fluid boundary

Superconductivity

Collective motion of loosely bound electron pairs at low temperatures





Mai-Linh Doan, Wikipedia

- **Superfluidity: Electrical resistance is zero**
- **No heat or spin transport in the superfluid**
- **Topological superconductor:**
  Theorized to possess a charge neutral surface fluid of unpaired "Majorana" fermions

Spaceballs the Movie

"New" idea: 3D Bulk topological superconductivity

Integer-valued winding number $\nu \in \mathbb{Z}$

Schnyder, Ryu, Furusaki, Ludwig 2008; Kitaev 2009

2D Majorana surface fluid, envelopes bulk

- **Transport properties?**
- **Stability? (Exposed crystal surface: disorder)**

Experimental realizations

- **Helium 3B (neutral topological superfluid)** **Volovik 1988**
- **$Cu_xBi_2Se_3$ (?) (doped top. insulator)** Fu and Berg 2010

Topological superconductor surface state Majorana fluid

For a 3D topological superconductor with bulk winding number ν, what do the Majorana surface states "look like?"

Spaceballs the Movie

Topological superconductor surface state Majorana fluid

A lot like graphene!

- **Unpaired surface Majorana fermion quasiparticles**
- **$|\nu| = 2k$ "colors," $k = (1,2,3,\ldots)$ (class CI)**

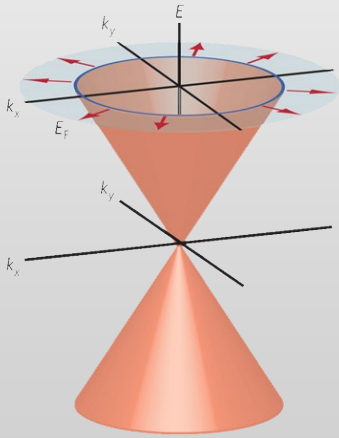
Low energy surface Andreev state Hamiltonian:

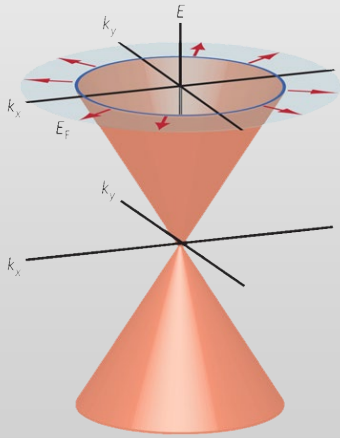
$$H = \int d^2\mathbf{r}\, \Psi^\dagger \left(-i\hat{\sigma}^1\partial_x - i\hat{\sigma}^2\partial_y\right)\Psi \equiv \Psi^\dagger \hat{h}\Psi$$

Majorana fermion carries Pseudospin (σ) and Color (κ) indices

$$\Psi_\kappa = \begin{bmatrix} c_{\uparrow,\kappa} \\ c^\dagger_{\downarrow,\bar{\kappa}} \end{bmatrix}$$

$$1 \le \kappa \le |\nu|$$





Schnyder, Ryu, Furusaki, Ludwig 08
Bernard, LeClair 02

- **"Anomalous" chiral symmetry (= *physical time-reversal*):** $-\hat{\sigma}^3\hat{h}\hat{\sigma}^3 = \hat{h}$

Topological superconductor surface state Majorana fluid

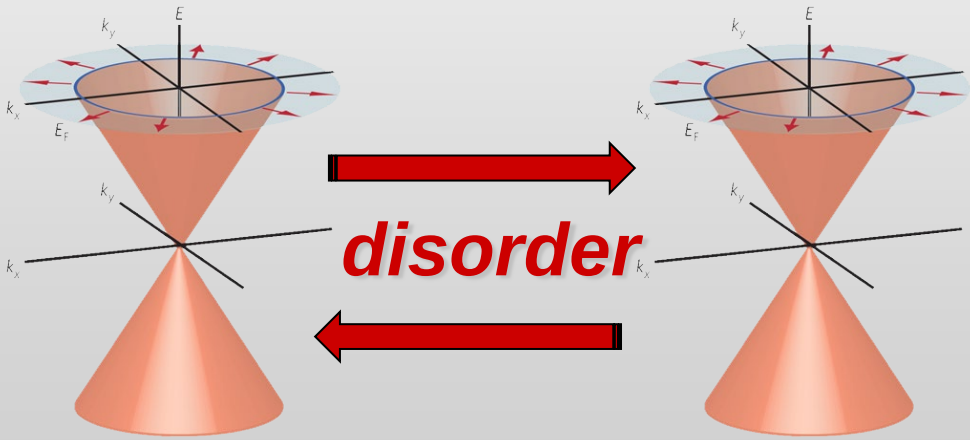
Effects of disorder

- **Junk is unavoidable at the surface!**
- ***Any*** **non-magnetic (time-reversal preserving) surface perturbation: intercolor vector potential $\hat{t}^i_\kappa \mathbf{A}_i(\mathbf{r})$!**

$$H = \int d^2\mathbf{r}\, \Psi^\dagger \left(-i\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nabla} + \mathbf{A}_i\cdot\boldsymbol{\sigma}\,\hat{t}^i_\kappa\right)\Psi = H_0 + \int d^2\mathbf{r}\, \left(J^i_\kappa \bar{A}_i + \bar{J}^i_\kappa A_i\right)$$

Sources of $\hat{t}^i_\kappa \mathbf{A}_i(\mathbf{r})$:

- **Impurities, vacancies**
- **External electric fields**
- **Edge, corner, dislocation potentials**



Schnyder, Ryu, Furusaki, Ludwig 08
Bernard, LeClair 02

"Quenched" 2+1-D QCD: Dirac fermions in a sea of frozen gauge fluctuations

Majorana fluid transport

Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. **Multiple scattering off of impurities (weak localization)**

Majorana fluid transport

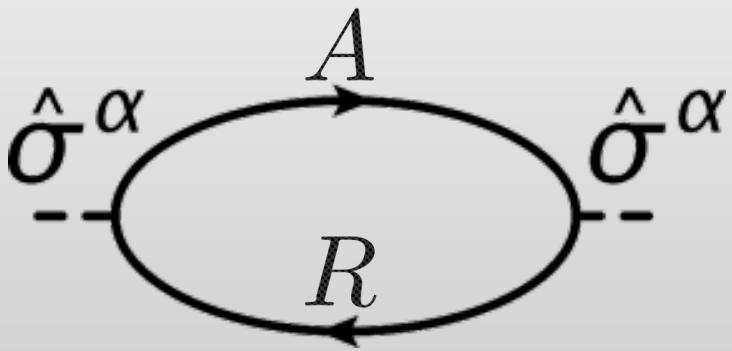
Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. Multiple scattering off of impurities (weak localization)

Kubo formula for dc spin conductivity:

$$\sigma = \frac{-1}{4\pi L^d} \int_{\mathbf{r}_1,\mathbf{r}_2} \mathrm{Re}\left\{ \mathrm{Tr}\left[\hat{\sigma}^\alpha \hat{G}^{(A)}(0;\mathbf{r}_1,\mathbf{r}_2)\hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r}_2,\mathbf{r}_1) - \hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r}_1,\mathbf{r}_2)\hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r}_2,\mathbf{r}_1) \right] \right\}$$



Components:
Retarded, advanced Green's functions

$$\hat{G}^{(R,A)}(\varepsilon;\mathbf{r}_1,\mathbf{r}_2) \equiv \langle \mathbf{r}_1 | \frac{1}{\varepsilon \pm i\eta - \hat{h}} | \mathbf{r}_2 \rangle$$

$$\hat{h} = -i\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nabla} + \mathbf{A}_i(\mathbf{r}) \cdot \boldsymbol{\sigma}\, \hat{t}^i_\kappa$$

Majorana fluid transport

Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. Multiple scattering off of impurities (weak localization)

Kubo formula for dc spin conductivity:

$$\sigma = \frac{-1}{4\pi L^d} \int_{\mathbf{r_1},\mathbf{r_2}} \mathrm{Re}\left\{ \mathrm{Tr}\left[\hat{\sigma}^\alpha \hat{G}^{(A)}(0;\mathbf{r_1},\mathbf{r_2}) \hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r_2},\mathbf{r_1}) - \hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r_1},\mathbf{r_2}) \hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r_2},\mathbf{r_1}) \right] \right\}$$

Anomalous form of time-reversal symmetry

Retarded, advanced interchangable: $-\hat{\sigma}^3 \hat{G}^{(A)}(\varepsilon;\mathbf{r_1},\mathbf{r_2}) \hat{\sigma}^3 = \hat{G}^{(R)}(-\varepsilon;\mathbf{r_2},\mathbf{r_1})$

$$\Rightarrow \quad \sigma = -\frac{1}{\pi} \lim_{\mathbf{r}\to\mathbf{r}'} \mathrm{Im}\left\{ \mathrm{Tr}\left[(\mathbf{r}-\mathbf{r}') \cdot \hat{\boldsymbol{\sigma}}\, \hat{G}^{(R)}(0;\mathbf{r},\mathbf{r}') \right] \right\} = \frac{|\nu|}{2\pi^2}$$

2+0-D Chiral anomaly

Universal spin, thermal conductivity, neglecting interactions

$$\sigma_s = \frac{|\nu|}{\pi h} \left(\frac{\hbar}{2}\right)^2, \qquad \kappa = \frac{|\nu|}{\pi h} \frac{\pi^2 k_B^2 T}{3}$$

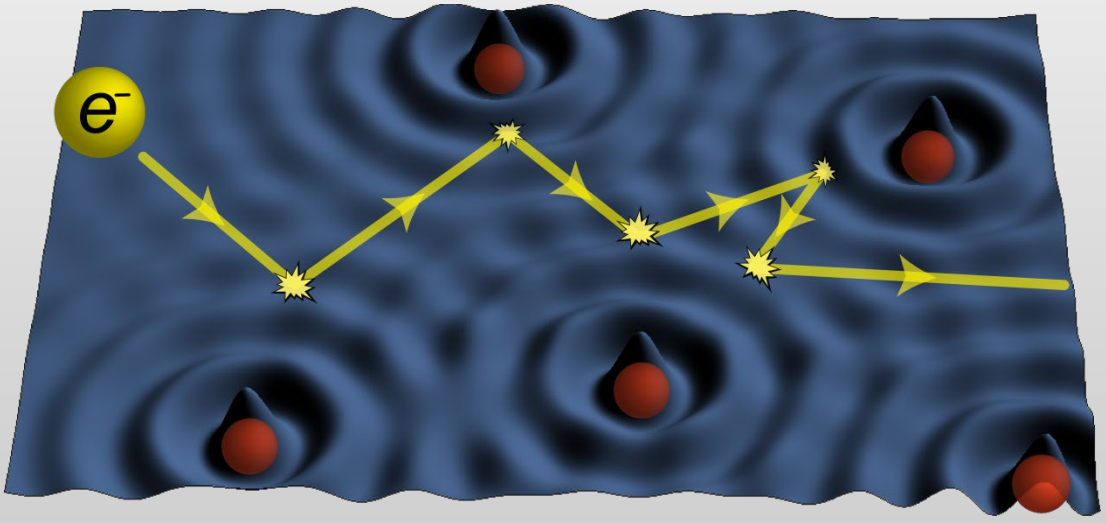
Ludwig, Fisher, Shankar, Grinstein (1994)
Tsvelik (1995)
Ostrovsky, Gornyi, Mirlin (2006)
Foster, Xie, and Chou (2014)

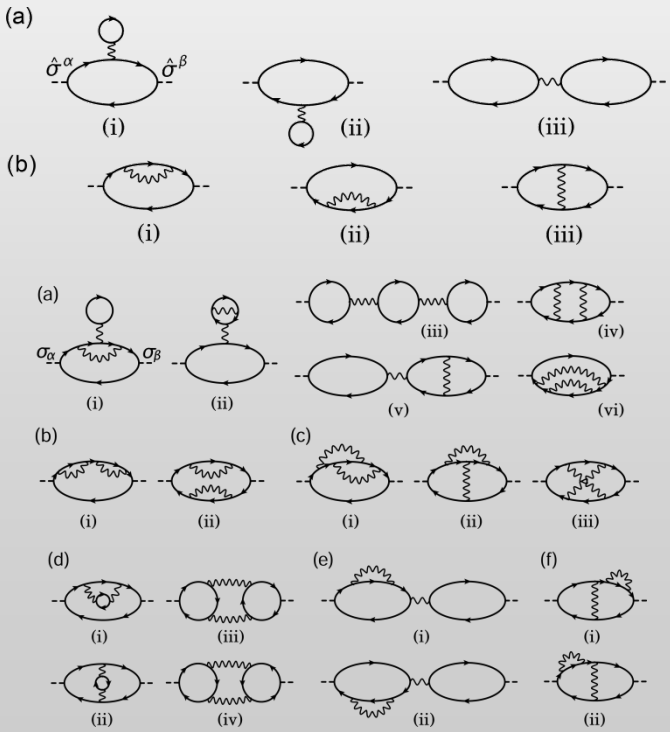
Majorana fluid transport

Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. **Multiple scattering off of impurities (weak localization)**
2. **Scattering off of impurity-induced density Friedel oscilations (Altshuler-Aronov corrections)**





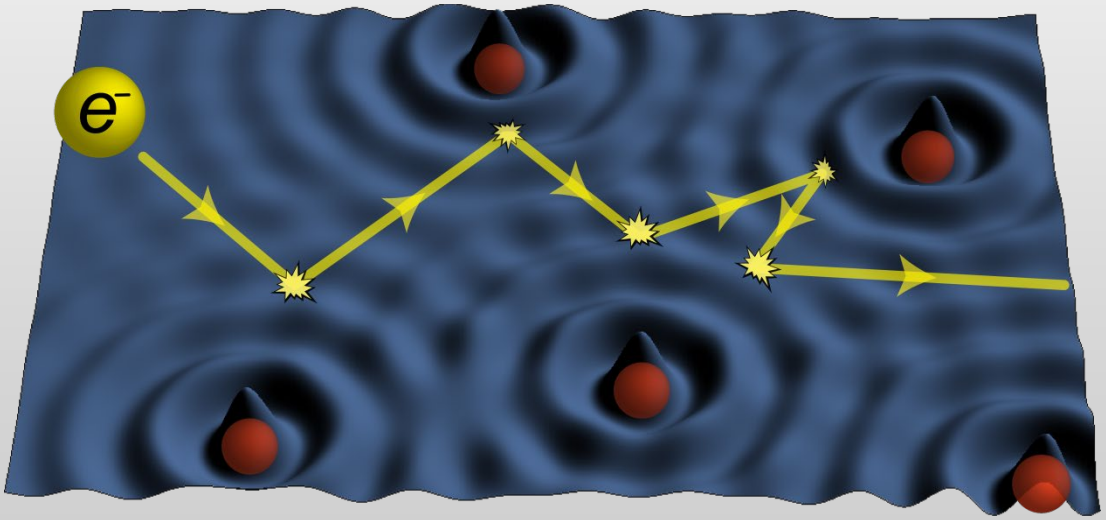
Altshuler and Aronov 1985
Aleiner, Altshuler, and Gershenson 1999
Zala, Narozhny, and Aleiner 2001

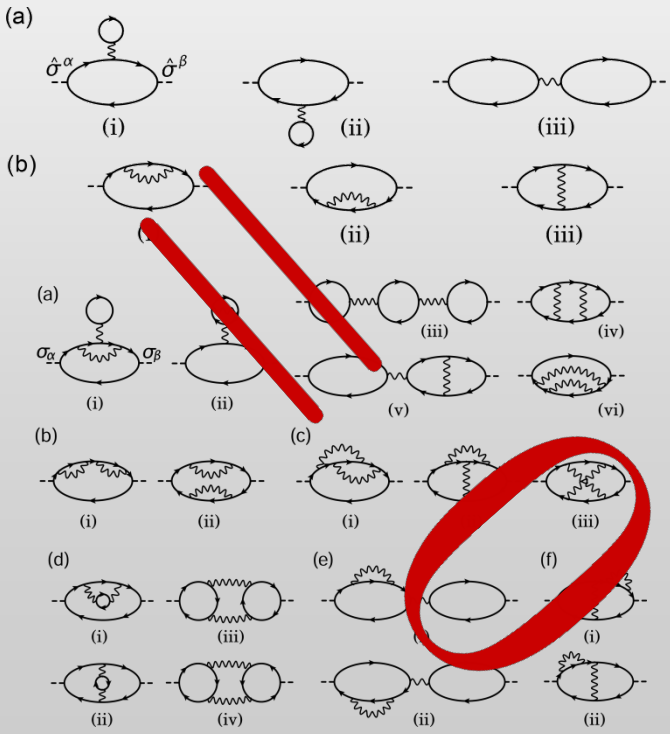
Majorana fluid transport

Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. **Multiple scattering off of impurities (weak localization)**
2. **Scattering off of impurity-induced density Friedel oscilations (Altshuler-Aronov corrections)**





Anomalous time-reversal symmetry:

$$-\hat{\sigma}^3 \hat{G}^{(A)}(\varepsilon; \mathbf{r_1}, \mathbf{r_2})\hat{\sigma}^3 = \hat{G}^{(R)}(-\varepsilon; \mathbf{r_2}, \mathbf{r_1})$$

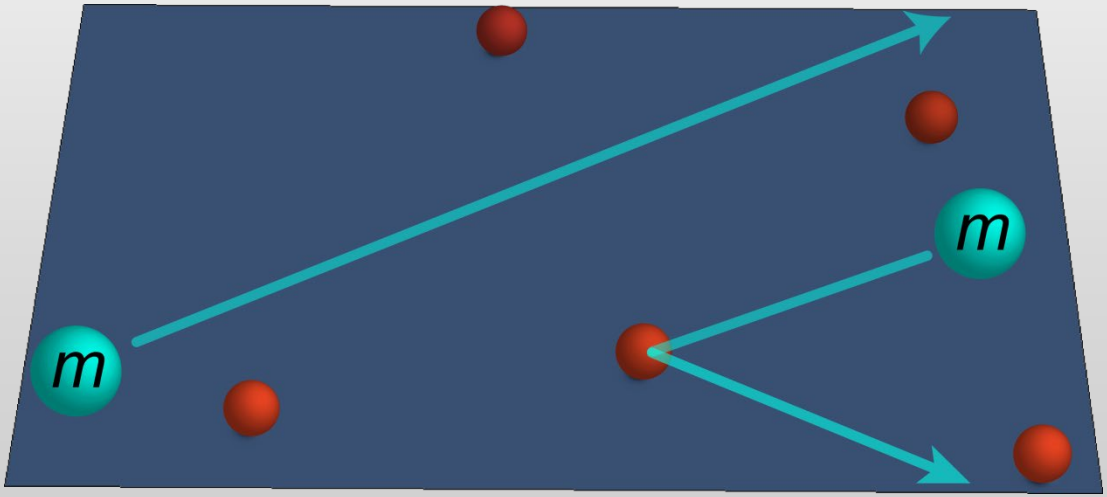
Xie, Chou, and Foster (2015)

Majorana fluid transport

Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. **Multiple scattering off of impurities (weak localization)**
2. **Scattering off of impurity-induced density Friedel oscilations (Altshuler-Aronov corrections)**



Anomalous time-reversal symmetry:
No Majorana "density" (mass, spin, color) can ripple (or become non-zero)!

Universal spin, thermal conductivities!

$$\sigma_s = \frac{|\nu|}{\pi h}\left(\frac{\hbar}{2}\right)^2$$

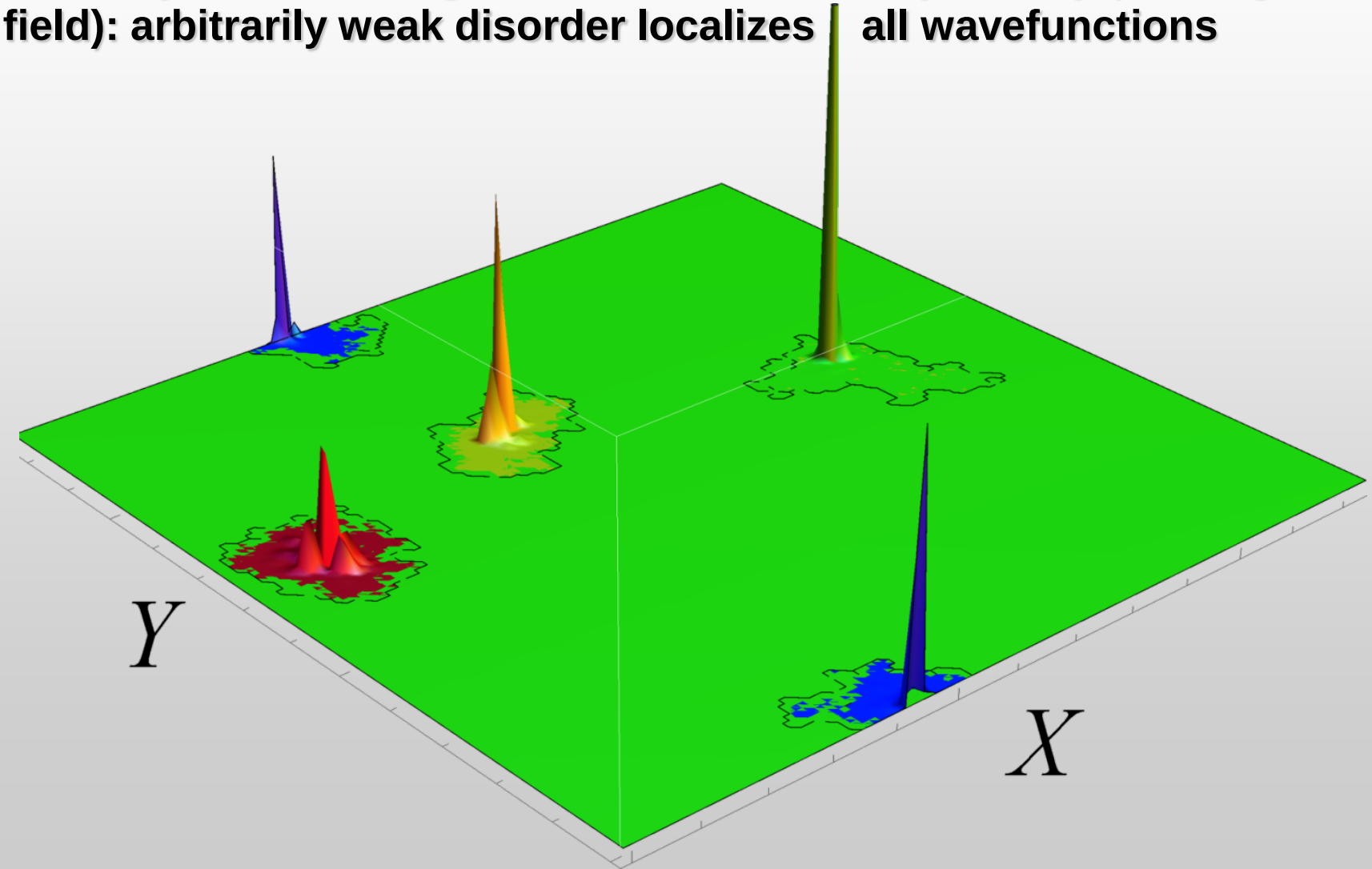
$$\kappa = \frac{|\nu|}{\pi h}\frac{\pi^2 k_B^2 T}{3}$$

Xie, Chou, and Foster (2015)

Majorana fluid wavefunctions

Universal transport: Disorder has no effect?

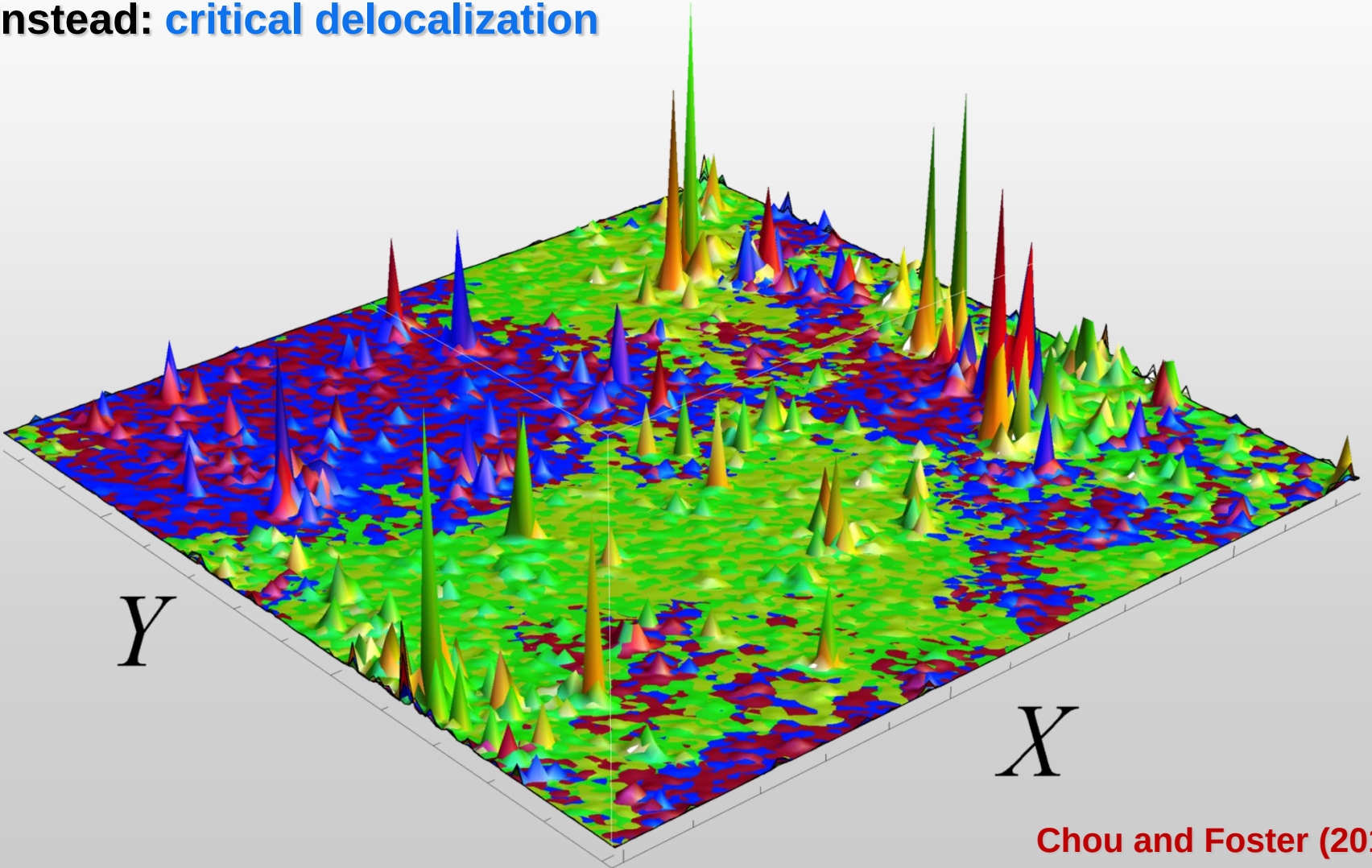
Ordinary 2D electron gas with time reversal symmetry (no magnetic field): arbitrarily weak disorder localizes all wavefunctions



Majorana fluid wavefunctions

Universal transport: Disorder has no effect?

Surface Majorana states cannot be localized (topological protection).
Instead: critical delocalization



Chou and Foster (2014)

Majorana fluid wavefunctions

2D Conformal field theory solution via *conformal embeddings*

Reviewed in (e.g.) J. Fuchs, *Affine Lie Algebras and Quantum Groups*

Class	Embedding	$\lvert\nu\rvert$	
CI	$\mathrm{SO}(4nk)_1 \supset \mathrm{Sp}(2n)_k \oplus \mathrm{Sp}(2k)_n$	$2k$	$(k \geq 1)$
AIII	$\mathrm{U}(nk)_1 \supset \mathrm{U}(n)_k \oplus \mathrm{SU}(k)_n$	k	$(k \geq 2)$
DIII	$\mathrm{SO}(nk)_1 \supset \mathrm{SO}(n)_k \oplus \mathrm{SO}(k)_n$	k	$(k \geq 3)$

Winding number |ν| = # colors

- "Fractionalization": Color sector "localizes"

Nersesyan, Tsvelik, Wenger 94

Wave functions are multifractal; exact spectra computed via CFT:

Exact Results

$$f(\alpha) = 2 - \frac{(\alpha - 2 - \theta_k)^2}{4\theta_k}$$

Class	$\lvert\nu\rvert$	θ_k	
CI	$2k$	$\frac{1}{2(k+1)}$	**Foster, Yuzbashyan 12**
AIII	$k \geq 1$	$\frac{k-1}{k^2} + \lambda_A$	Mudry, Chamon, Wen 96 Caux, Kogan, Tsvelik 96
DIII	$k \geq 3$	$\frac{1}{k-2}$	**Foster, Xie, Chou 14**

Numerical tests: Critical DOS, multifractal scaling

Minimal case: 2 valley Dirac (Classes CI and AIII)

CFT predictions:

- **Global density of states**

$$\nu(\varepsilon) \sim |\varepsilon|^{\eta}, \quad \eta = \frac{1-4\lambda_A}{7+4\lambda_A}$$

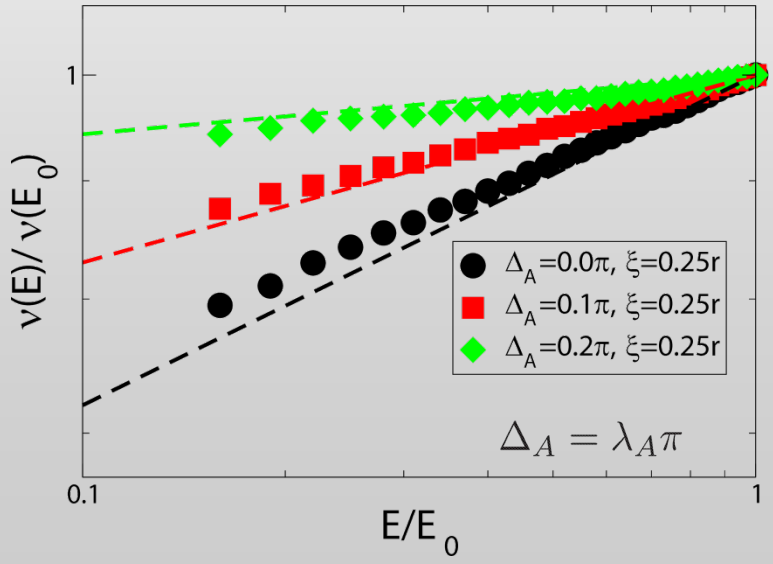
- **Multifractal spectrum**

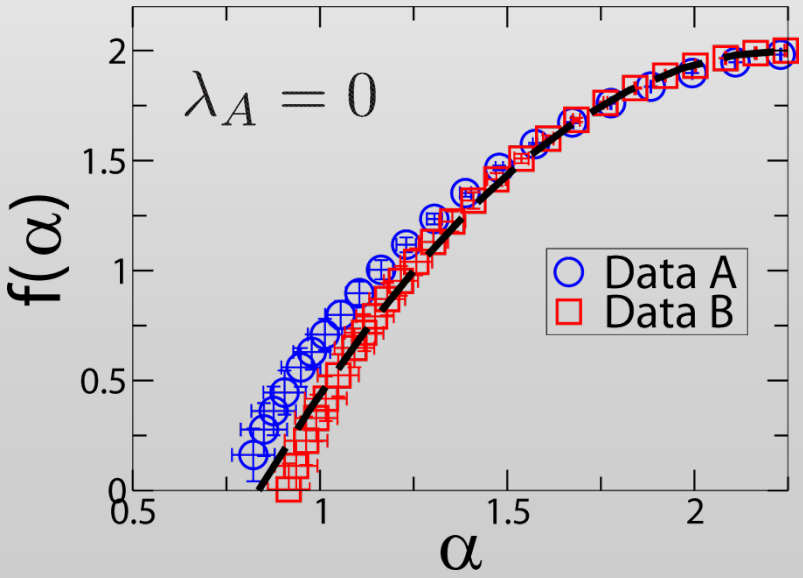
$$f(\alpha) = 8\frac{(\alpha_+ - \alpha)(\alpha - \alpha_-)}{(\alpha_+ - \alpha_-)^2}, \quad \alpha_\pm = (\sqrt{2} \pm \sqrt{\theta_2})^2$$

Numerical scheme:

Momentum-space disordered Dirac fermion (avoids fermion doubling)

Bardarson, Tworzydlo, Brouwer, Beenakker 07
Nomura, Koshino, Ryu 07





Chou and Foster 2014

Topological protection? Disorder and interactions

Extended, multifractal surface states:
No Anderson localization = topological protection! **…BUT**

Add generic, *weak* **interparticle interactions**, consistent with bulk symmetries [time-reversal, spin SU(2) for CI]

$$H_I = U \int d^2\mathbf{r}\, \Psi^\dagger_\alpha \Psi_\beta \Psi^\dagger_\gamma \Psi_\delta$$

Physical picture: Chalker scaling, multifractality, and interactions

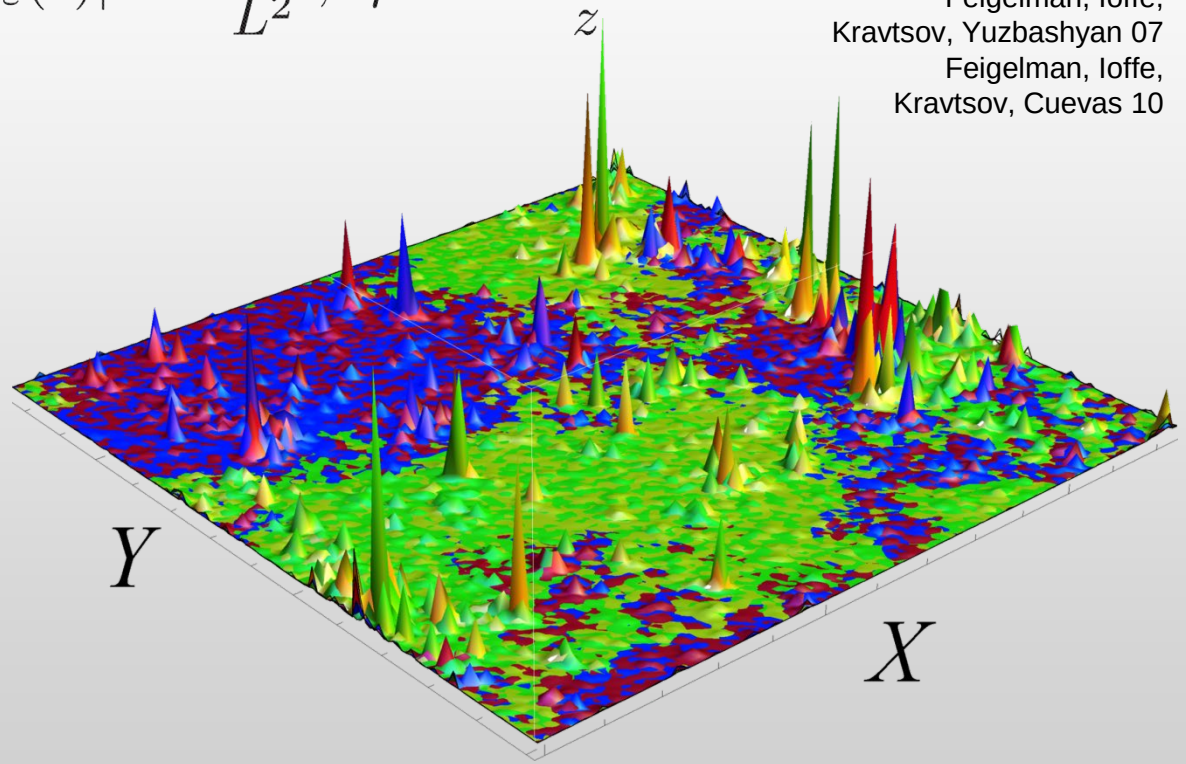
- **Chalker scaling: Overlapping peaks and valleys in multifractal eigenstates with nearby energies**

$$\lim_{L\to\infty} \int d^2\mathbf{r}\, |\psi_0(\mathbf{r})|^2 |\psi_\varepsilon(\mathbf{r})|^2 \sim \frac{\varepsilon^{-\mu}}{L^2}, \quad \mu = \frac{2-\tau(2)}{z}$$

Chalker, Daniell 88
Chalker 90
Cuevas, Kravtsov 07

Feigelman, Ioffe, Kravtsov, Yuzbashyan 07
Feigelman, Ioffe, Kravtsov, Cuevas 10

Probability peaks in *different* wavefunctions tend to cluster



Chou and Foster (2014)

Physical picture: Chalker scaling, multifractality, and interactions

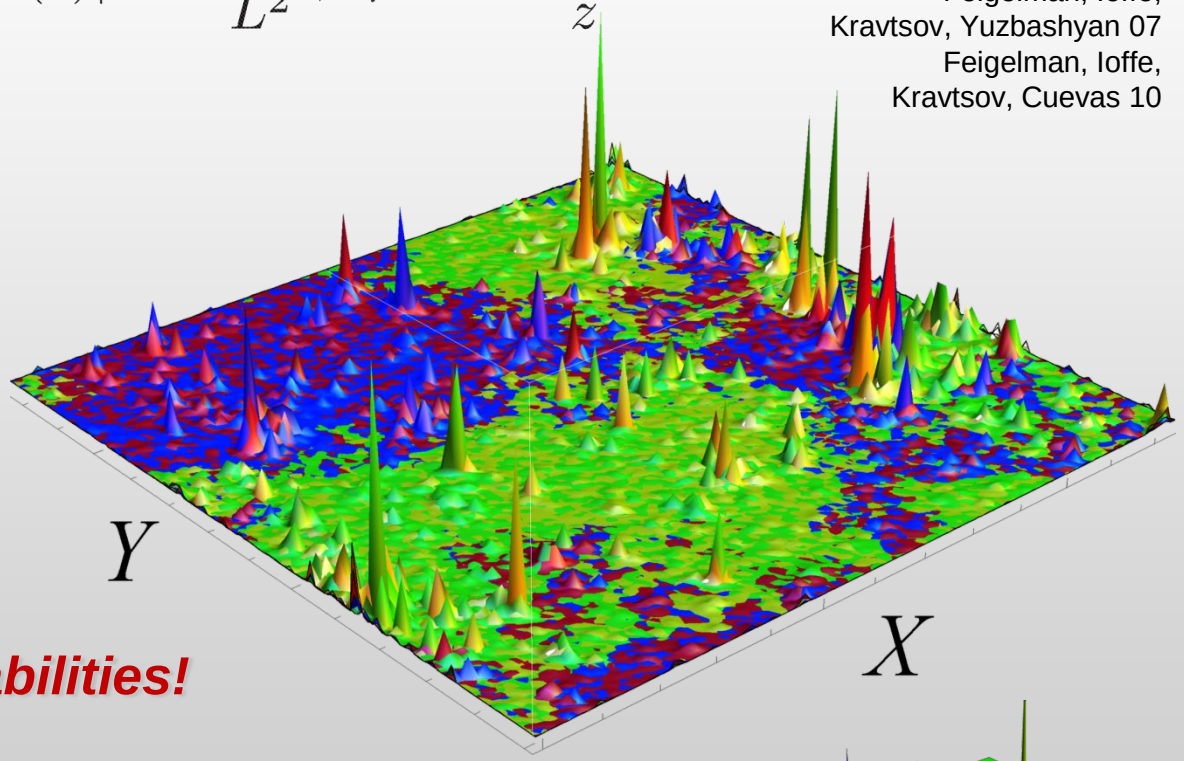
- **Chalker scaling: Overlapping peaks and valleys in multifractal eigenstates with nearby energies**

Chalker, Daniell 88
Chalker 90
Cuevas, Kravtsov 07

Feigelman, Ioffe, Kravtsov, Yuzbashyan 07
Feigelman, Ioffe, Kravtsov, Cuevas 10

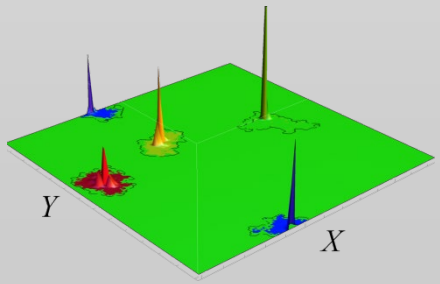
$$\lim_{L\to\infty}\int d^2\mathbf{r}\,|\psi_0(\mathbf{r})|^2|\psi_\varepsilon(\mathbf{r})|^2 \sim \frac{\varepsilon^{-\mu}}{L^2}, \quad \mu = \frac{2-\tau(2)}{z}$$

Probability peaks in *different* wavefunctions tend to cluster



- **Why scaling theory of localization works**
- **Enhances interaction matrix elements *–instabilities!***
- **Anderson insulator: No overlap for nearby energies**

$$|\psi_\varepsilon(\mathbf{r})|^2|\psi_{\varepsilon'}(\mathbf{r})|^2 \sim 0, \quad 0 < |\varepsilon-\varepsilon'| \ll \delta_l$$

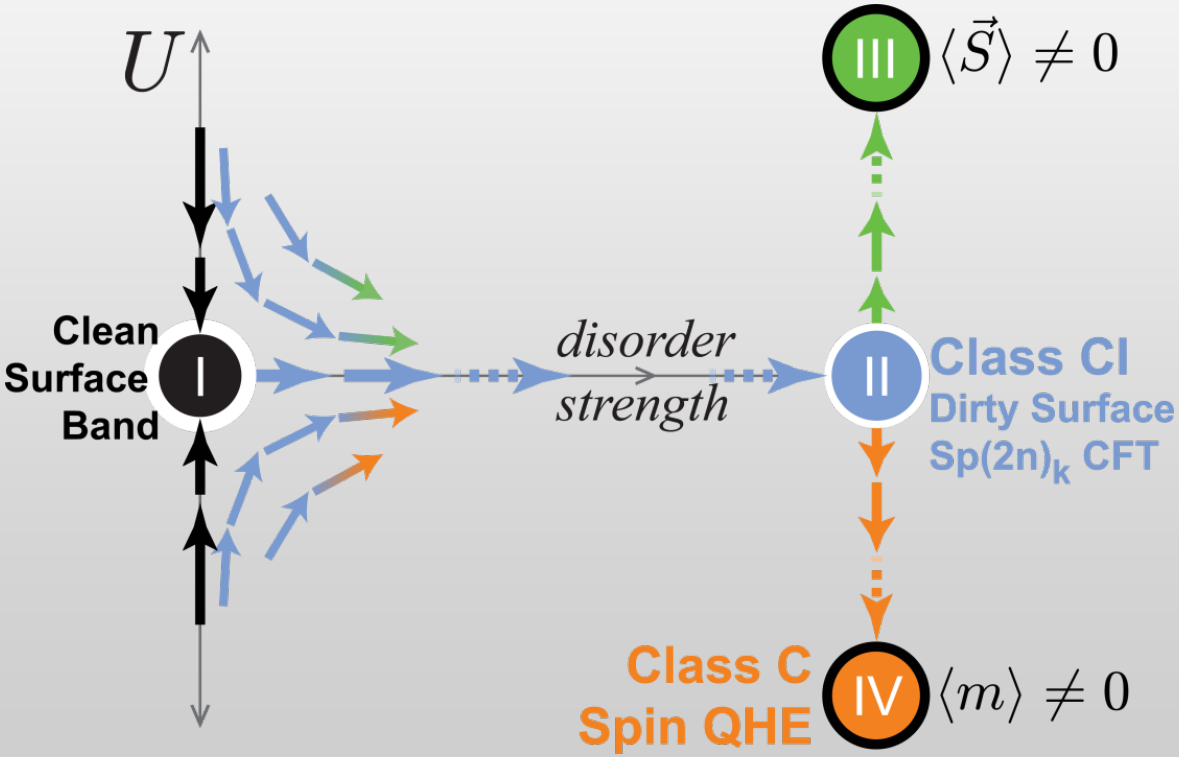


Weak disorder and interactions can sabotage topological protection!

- **Result: Not always protected. Even weak disorder and weak interactions can destroy some surface states**

Class CI Topological superconductors: Majorana surface fluid always unstable for any disorder, interactions, winding number

Foster, Yuzbashyan (2012)

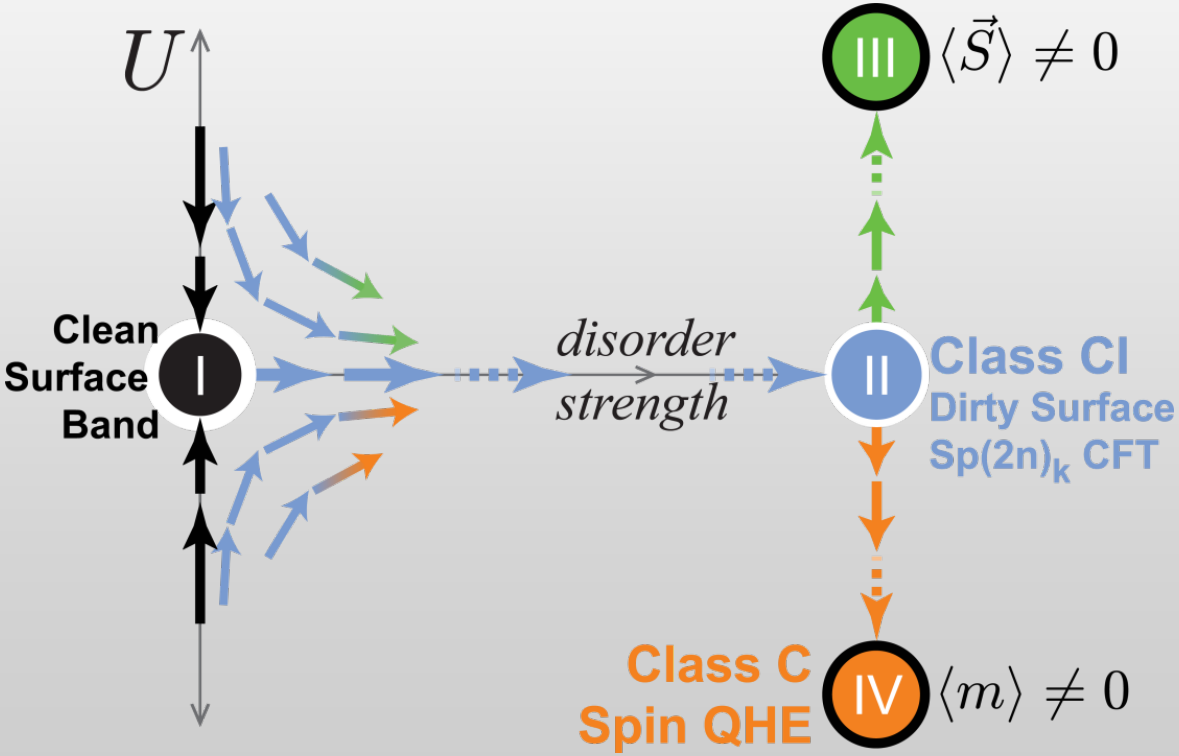


Weak disorder and interactions can sabotage topological protection!

- **Result: Not always protected. Even weak disorder and weak interactions can destroy some surface states**

Class CI Topological superconductors: Majorana surface fluid always unstable for any disorder, interactions, winding number

Foster, Yuzbashyan (2012)



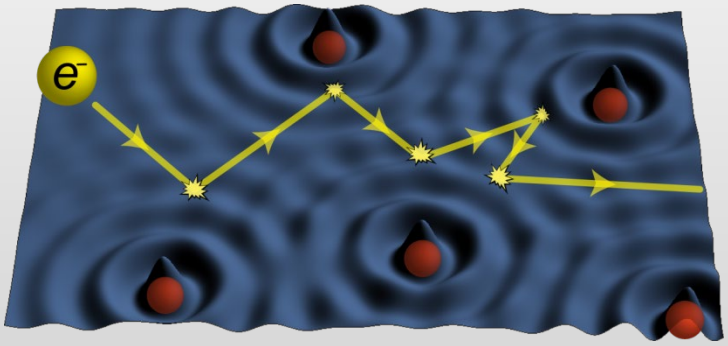
Classes AIII, DIII: Stable surface states

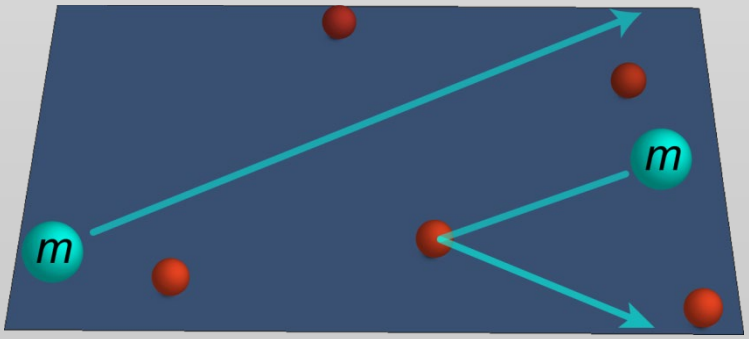
Foster, Xie, Chou (2014)

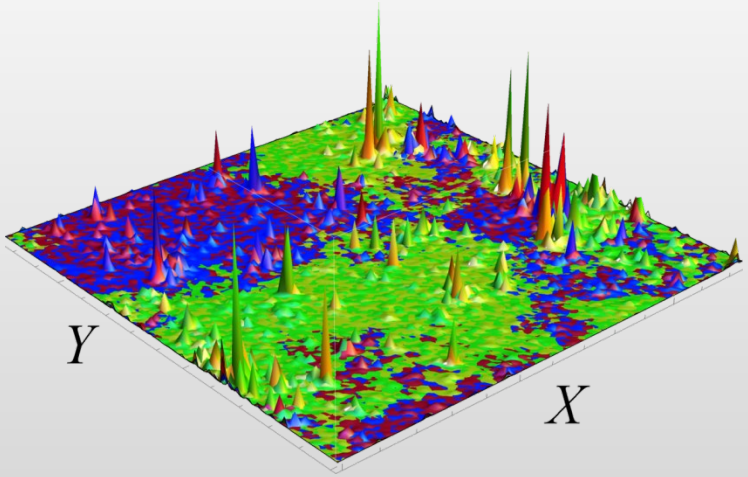
Summary

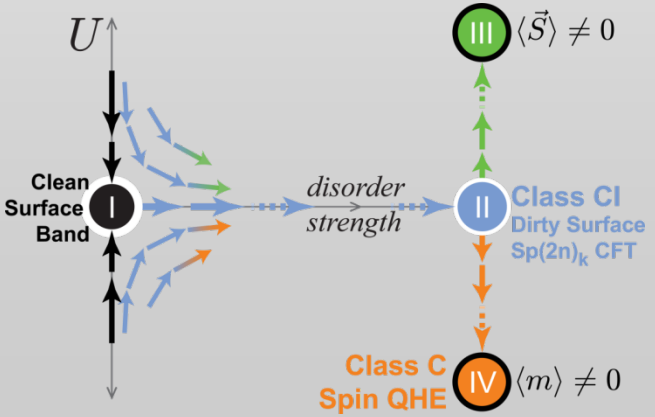
2D Majorana liquid theory

- Surface states of a bulk topological superconductor
- Universal transport coefficients encode bulk winding number
- Combined effects of disorder and interactions can lead to instabilities









Summary

2D Majorana liquid theory

- Surface states of a bulk topological superconductor
- Universal transport coefficients encode bulk winding number
- Combined effects of disorder and interactions can lead to instabilities

3D Topological superconductivity: close analog of the integer quantum Hall effect

- Is there a fractional analog? (Bulk with topological order; gapless surface fluid with fractionalized transport coefficients)
- What about gapless (nodal) "topological" superconductor surface states?

Sato 2006
Beri 2010
Schnyder and Ryu 2011
Matsuura, Chang, Schnyder, Ryu 2013
Zhao and Wang 2013

- Materials beyond $Cu_xBi_2Se_3$?!