Interaction-mediated surface state instability in dirty topological superconductors

Matthew S. Foster\textsuperscript{1} and Emil A. Yuzbashyan\textsuperscript{2}

\textsuperscript{1} Rice University, \textsuperscript{2} Rutgers University

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Topological insulators: A knotty state of affairs

Topologically non-trivial bulk

“Unwinding” edge/surface state

Topologically trivial vacuum

3D topological insulator theory:
- Fu, Kane, and Mele 2007
- Moore and Balents 2007
- Qi, Hughes, and Zhang 2008
- R. Roy 2009
Topological surface state on Bi$_2$Se$_3$ = “$\frac{1}{4}$ of graphene”

- “Holography”
  *Unpaired* 2D Dirac fermion in Time-Reversal Invariant system only allowed as surface of 3D bulk
- Half-integer Quantum Hall effect (parity anomaly)
- Quantized magnetoelectric coupling, axion electrodynamics…

Fu, Kane, and Mele 2007, Moore and Balents 2007, Fu and Kane 2007; Qi, Hughes, and Zhang 2008, R. Roy 2009
Philip Anderson, 1958:

A sufficiently strong random potential $V(r)$ exponentially localizes all single particle states (at a given energy).

Weak disorder:  
**Extended states**

$$|\psi|^2(r)$$

$V(r)$

Strong disorder:  
**Localized states**

Characterized by localization length $\xi_{\text{loc}}$

$$|\psi|^2(r)$$

$V(r)$
Anderson Localization: Basics

Weak disorder: Extended states

\[
|\psi|^2(r) \quad \rightarrow (Dirty) \text{ Metal!}
\]

Strong disorder: Localized states

Characterized by localization length \( \xi_{\text{loc}} \)

\[
|\psi|^2(r) \quad \rightarrow \text{Anderson Insulator!}
\]

Philip Anderson, 1958:

A sufficiently strong random potential \( V(r) \) exponentially localizes all single particle states (at a given energy)
The multifractal spectrum $\tau(q)$: Probe for quantum interference

- “Topologically-protected surface states”
  Protection from Anderson localization!

- Localized or not? Inverse Participation Ratio (IPR)

$$P_q \equiv \int_{L^d} |\psi(\mathbf{r})|^2 q \, d^d\mathbf{r} \sim \left(\frac{a}{L}\right)^{\tau(q)}$$

- As a probe of wavefunction (de)localization:

  (a) Extended plane wave state

  $$\tau(q) = d(q - 1)$$

  (b) Exponentially localized state

  $$\tau(q) \sim 0, \quad L \gg \xi_{\text{loc}}$$

Bardarson, Tworzydło, Brouwer, Beenakker 2007
Nomura, Koshino, Ryu 2007
Ryu, Mudry, Obuse, Furusaki 2007
Ostrovsky, Gornyi, Mirlin 2007
Critical wavefunctions at a delocalization transition

Example: Anderson Metal to Insulator transition in 3D

- ψ exhibits **multifractal scaling**
  (neither localized nor extended, nor a simple fractal)
  
  Wegner 1980; Castellani and Peliti 1986

- Multifractality encoded in **nonlinearity**
  of the τ(q) spectrum

Vasquez, Rodriguez, Roemer 2008
Wavefunction multifractality at a delocalization transition

**Example:** Multifractal spectra at the Integer Quantum Hall Plateau Transition

- States away from LL center are Anderson localized
- **Delocalized states at the transition** (non-zero longitudinal resistance/conductance)

Prange 1987
Wavefunction multifractality at a delocalization transition

Example: Multifractal spectra at the Integer Quantum Hall Plateau Transition

- States away from LL center are Anderson localized
- **Delocalized states at the transition** (non-zero longitudinal resistance/conductance)

\[ |\psi(r)|^2 \]

for an extended state (numerics)
Wavefunction multifractality: plateau transition

- $\tau(q)$ is both **self-averaging** and **universal**

- $f(\alpha) \equiv \alpha q - \tau(q)$ “singularity spectrum”

Chamon, Mudry, Wen 1996; Mirlin and Evers 2000; Obuse, Subramaniam, Furusaki, Gruzberg, Ludwig 2008; Evers, Mildenberger, Mirlin 2008

**Graphs:**

- **Real space Hamiltonian**
  - Pook and Janssen 1991

- **Tight-binding model**
  - Huckestein 1995
Experiment? On *metallic side*, can use LDOS $\rho(\epsilon,r)$ instead:

$$P_q = \frac{\int_{L^d} \rho^q(\epsilon,r) \, d^d r}{\left( \int_{L^d} \rho(\epsilon,r) \, d^d r \right)^q} \sim \left( \frac{a}{L} \right)^\tau(q)$$

Multifractal spectra for topological insulator surface states

Foster PRB 2012

Multifractal LDOS fluctuations in GaMnAs

Richardella, Roushan, Mack, Zhou, Huse, Awschalom, and Yazdani 2010
Schnyder, Ryu, Furusaki, Ludwig 2008, 2010; Kitaev 2009:

- In any number $d$ of spatial dimensions, 5 symmetry classes of topological materials, topologically-protected surface states
- These are a subset of 10 random matrix classes (also used in Anderson localization)

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Green: 3D $Z_2$ Topological Insulator ($\text{Bi}_2\text{Se}_3$, etc)

Red: 3D Topological Superconductors (TSC)

Altland and Zirnbauer 1997
Bernard and LeClair 2002
Spin singlet TSC: class $\text{Cl} [\text{spin SU(2) symmetry}]$

**3D Lattice model**

\[
H_{\text{bulk}} = \sum_{i,j} \left[ t_{ij} c_{i,s}^\dagger c_{j,s} + \Delta_{ij} c_{i,\uparrow}^\dagger c_{j,\downarrow}^\dagger + \text{H.c.} \right]
\]

- “Haldane-Kane-Mele” type model
- Spin $\frac{1}{2}$ electrons
- No spin-orbit coupling
- $nn$ hopping on the diamond lattice at half-filling
- $nnn$ d-wave pairing—3D BdG quasiparticle nodes

Read and Green 2000; non-trivial topology $\iff$ non-$s$ wave pairing

Bulk not fully gapped!

Not yet a topological superconductor!
Spin singlet TSC: class CI [spin SU(2) symmetry]

3D Lattice model

- **Bulk** quasiparticle mass gap:
  
  (a) sublattice staggered chemical potential $\mu_S$ (trivial)
  
  (b) nnn hopping $t'$ (“topological”)

- Winding number:

  \[
  \nu[q] \equiv \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho} \text{Tr} \left[ (q^{-1} \partial_\mu q)(q^{-1} \partial_\nu q)(q^{-1} \partial_\rho q) \right] 
  \in \{-2, 0, 2\}
  \]

Schnyder, Ryu, Furusaki, Ludwig 2008

Schnyder, Ryu, Ludwig 2009
Spin singlet TSC: class CI [spin SU(2) symmetry]

- Topologically-protected, gapless surface state (Bogoliubov) quasiparticles
- $|\mathcal{V}| = 2k$ valleys, $k = (1, 2, 3, \ldots)$; SRL model has $k = 1$

Low energy surface Andreev state Hamiltonian:

$$H = \int d^2 r \, \Psi^\dagger \left( -i \hat{\sigma} \cdot \nabla \right) \Psi$$

- Surface “Majorana” (Andreev bound state) band fermion

$$\Psi_v = \begin{bmatrix} c_{\uparrow, v} \\ c_{\downarrow, v} \end{bmatrix}$$
Spin singlet TSC: class CI [spin SU(2) symmetry]

- Topologically-protected, gapless surface state (Bogoliubov) quasiparticles
- \(|\nu| = 2k\) valleys, \(k = (1,2,3,\ldots)\); SRL model has \(k = 1\)

Low energy surface Andreev state Hamiltonian:

\[
H = \int d^2 r \, \Psi^\dagger (-i \hat{\sigma} \cdot \nabla) \Psi \equiv \Psi^\dagger \hat{h} \Psi
\]

- “Topological” chiral symmetry (\(= \text{physical time-reversal}\)):
  \(-\hat{\sigma}^3 \hat{h} \hat{\sigma}^3 = \hat{h}\)

- “Particle-hole” symmetry \(= \text{spin SU(2) invariance}\):
  \(-\hat{\sigma}^1 \hat{\kappa}^2 \hat{h}^\dagger \hat{\sigma}^1 \hat{\kappa}^2 = \hat{h}\)
Spin singlet TSC: class CI [spin SU(2) symmetry]

- Topologically-protected, gapless surface state (Bogoliubov) quasiparticles
- \( |V| = 2k \) valleys, \( k = (1,2,3,...) \); SRL model has \( k = 1 \); \( n \) replicas

Low energy surface Andreev state Hamiltonian:

\[
H = \int d^2r \, \bar{\Psi} (-i \hat{\sigma} \cdot \nabla) \Psi
= - \int d^2r \, (L \bar{\delta} \mathcal{L} + R \partial \mathcal{R})
\]

\[
\mathcal{L} = \begin{bmatrix}
\Psi_1 \\
-\hat{\kappa}^2 \left( \Psi^\dagger_2 \right)^T
\end{bmatrix} \rightarrow \mathcal{L}_{s,v,a}
\]

Invariant under chiral (independent left and right) transformations:

- Spin SU(2) rotations
  \[
  \mathcal{L}_{s,v,a} \rightarrow \left[ \exp(-i \hat{S} \cdot \phi) \right]_{s,s',v,a} \mathcal{L}_{s',v,a}
  \]
- Valley Sp(2 k) rotations
  \[
  \mathcal{L}_{s,v,a} \rightarrow \left[ \exp(-i \hat{\eta} \theta \eta) \right]_{v,v',a} \mathcal{L}_{s,v',a}
  \]
- SO(n) replica space rotations
  \[
  \mathcal{L}_{s,v,a} \rightarrow \left[ \exp(-i \hat{T} \xi \gamma) \right]_{a,a',v} \mathcal{L}_{s,v,a'}
  \]
Spin singlet TSC: class CI [spin SU(2) symmetry]

- Topologically-protected, gapless surface state (Bogoliubov) quasiparticles
- $|V| = 2k$ valleys, $k = (1,2,3,...)$; SRL model has $k = 1$; $n$ replicas

Conformal Field Theory: $SO(4 n k)_1$ (free fermions)

$$H = \int d^2 r \, \Psi^\dagger (-i \hat{\sigma} \cdot \nabla) \Psi$$
$$= - \int d^2 r \, (L \, \bar{\partial} L + R \, \partial R)$$

$$\mathcal{L} \equiv \begin{bmatrix} \Psi_1^\dagger \\ -\hat{\kappa}^2 (\Psi_2^\dagger)^T \end{bmatrix} \rightarrow \mathcal{L}_{s,v,a}$$

Invariant under chiral (independent left and right) transformations:

- Combined spin x valley x replica SO(4 n k) rotations

$$\mathcal{L} \rightarrow \exp \left( -i \hat{S}^\Sigma \, t^i_\kappa \, T^\gamma_R \, \Sigma^i \gamma \right) \mathcal{L}$$
Spin singlet TSC: class CI [spin SU(2) symmetry]

- Topologically-protected, gapless surface state (Bogoliubov) quasiparticles
- TRI disorder \(\rightarrow\) multifractal surface states \((n\) replicas, \(2k\) valleys)

Conformal Field Theory: \(\text{SO}(4n k)_1 \supset \text{Sp}(2n)_k \oplus \text{Sp}(2k)_n\)

\[
H = \int d^2r \bar{\Psi} \left( -i \sigma \cdot \nabla + A_i \cdot \sigma \hat{\tau}^i_{\kappa} \right) \Psi = H_0 + \int d^2r J^i_{\kappa} \bar{\Phi}_i + \bar{J}^i_{\kappa} \Phi_i
\]

- “Topological” chiral symmetry (= physical time-reversal):
- “Particle-hole” symmetry [= spin \(\text{Sp}(2)\) invariance]:

\[
-\hat{\sigma}^3 \hat{\hbar} \hat{\sigma}^3 = \hat{\hbar} \\
-\hat{\sigma}^1 \hat{\kappa}^2 \hat{\hbar}^T \hat{\sigma}^1 \hat{\kappa}^2 = \hat{\hbar}
\]
Spin singlet TSC: class CI [spin SU(2) symmetry]

- Topologically-protected, gapless surface state (Bogoliubov) quasiparticles
- TRI disorder \( \Rightarrow \) multifractal surface states \((n\text{ replicas, } 2k\text{ valleys})\)

Conformal Field Theory: \( \text{SO}(4nk)_1 \supset \text{Sp}(2n)_k \oplus \text{Sp}(2k)_n \)

- LDOS multifractal spectrum [\( \text{Sp}(2n)_k\) primary fields]

\[
\tilde{\tau}(q) = 2(q - 1) + \Delta_q - q\Delta_1 = (q - 1) \left[ 2 - \frac{1}{2(k+1)} q \right]
\]

\( k = 1:\)
- Nersesyan, Tsvelik, Wenger 1994
- Mudry, Chamon, Wen 1996
- Caux, Kogan, Tsvelik 1996

\( k > 1:\)
- Bernard and LeClair 2002
- Schnyder, Ryu, Furusaki, Ludwig 2008

- Extended, multifractal surface states: No Anderson localization = topological protection!
Spin singlet TSC: class CI [spin SU(2) symmetry]

- Topologically-protected, gapless surface state (Bogoliubov) quasiparticles
- TRI disorder → multifractal surface states ($n$ replicas, $2k$ valleys)

Conformal Field Theory: $SO(4n_k) \supset Sp(2n)_k \oplus Sp(2k)_n$

- LDOS multifractal spectrum [$Sp(2n_k)$ primary fields]

$$\tilde{\tau}(q) = 2(q - 1) + \Delta_q - q\Delta_1 = (q - 1) \left[ 2 - \frac{1}{2(k+1)} q \right]$$

**Different from Z2**

**Topological insulators:**

- Majorana surface band delocalized only at the chemical potential (which lies in bulk gap)
- At non-zero energy, crossover to orthogonal class

**All other states localized! (~IQHP)**

- Loc. length $\zeta(\varepsilon) \sim |\varepsilon|^{-1/z}$; $z = \frac{4k+3}{2(k+1)}$

$k = 1$: Nersesyan, Tsvelik, Wenger 1994
Mudry, Chamon, Wen 1996
Caux, Kogan, Tsvelik 1996

$k > 1$: Foster and Yuzbashyan 2012
Add generic, weak interparticle interactions, consistent with bulk symmetries [time-reversal, spin SU(2), and valley symmetry*]

\[ H_I = U \int d^2 r \, \Psi_\alpha^\dagger \Psi_\beta \Psi_\gamma^\dagger \Psi_\delta \]
Clean limit: DOS dimension $\Delta_1 = d - z$ determines relevance of short-ranged interactions

$$\frac{dU}{dl} = (\Delta_1 - \Delta_2^{(U)}) U = -\Delta_1 U + O(U^2), \quad \Delta_2^{(U)} = 2\Delta_1$$

Clean Dirac: $\Delta_1 = 1 \implies$ interactions irrelevant!

$$H_I = U \int d^2 r \; \Psi_\alpha^\dagger \Psi_\beta \Psi_\gamma^\dagger \Psi_\delta$$
**Clean limit:** DOS determines relevance of short-ranged interactions

\[ \frac{dU}{dl} = (\Delta_1 - \Delta_2^{(U)})U = -\Delta_1 U + O(U^2), \quad \Delta_2^{(U)} = 2\Delta_1 \]

**Clean Dirac:** \( \Delta_1 = 1 \) → interactions irrelevant!

**Dirty case:**

- \( \Delta_1 \equiv \) scaling dimension of disorder-averaged LDOS
- \( \Delta_2^{(U)} \equiv \) scaling dimension of disorder-averaged interaction

**Constraint:** \( \Delta_2^{(U)} \geq \Delta_2 \)

- \( \Delta_2 \equiv \) scaling dimension of second LDOS moment
**Clean limit:** DOS determines relevance of short-ranged interactions

\[
\frac{dU}{dl} = (\Delta_1 - \Delta_2^{(U)})U = -\Delta_1 U + O(U^2), \quad \Delta_2^{(U)} = 2\Delta_1
\]

**Clean Dirac:** \( \Delta_1 = 1 \) \( \rightarrow \) interactions irrelevant!

**Dirty case:**

- \( \Delta_1 \equiv \) scaling dimension of disorder-averaged LDOS
- \( \Delta_2^{(U)} \equiv \) scaling dimension of disorder-averaged interaction

Under suitable circumstances \( \Delta_2^{(U)} = \Delta_2 \) (for particular interactions)

- \( \Delta_2 \equiv \) scaling dimension of second LDOS moment

**Convexity property for a multifractal extended surface state:**

\[
\Delta_2 < 2\Delta_1 \quad \text{(independent dimensions!)}
\]

\[ \therefore \text{ Wavefunction multifractality can amplify short-ranged interactions! } \]
For disordered TSC surface system, expect important interactions are

- Cooper pairing of surface quasiparticles (time-reversal invariance)
- Spin exchange (spin is conserved = hydrodynamic mode)

One particular combination turns out to be most important:

\[ H_I = U \int d^2r \left[ m(r) m(r) - 4\tilde{S}(r) \cdot \tilde{S}(r) \right] + \ldots \text{(irrelevant)} \]

Order parameters break time-reversal:

- **Spin polarization** \( \tilde{S} = c^\dagger \frac{\sigma_i}{2} c \)
- **Imaginary s-wave pairing mass** \( m \sim -ic^\dagger c_\downarrow + ic_\downarrow c_\uparrow; \quad \tilde{\sigma}^{xy}_s = k \text{sgn}(m) \)

**Class C Spin QHE**
Aside: Topological “Half-(twice)-integer” Spin QHE

“Regular” 2D Spin QHE

Senthil, Marston, Fisher 1999

\[ \sigma_{xy} = 2 \]

\[ \sigma_{xy} = 0 \]

2D Sample Edge

Vacuum

“Half”-Spin QHE (Top SC surface)

Gruzberg, Ludwig, Read 1999

\[ \tilde{\sigma}_{xy} = -1 \]

\[ \tilde{\sigma}_{xy} = +1 \]

Class C SQHE Phase Transition

Classical Percolation CFT

Ginzburg-Landau Hamiltonian

\[ \mathcal{H} = V t \bar{c} \sigma \gamma \gamma' c' + \mathcal{D}(x) \mathbf{c}^\dagger \mathbf{c} + \mathcal{D}(y) \mathbf{c} \mathbf{c}^\dagger \]

\[ \mathcal{D}_0 + \frac{i}{2} \mathcal{D}_1 \sigma \gamma \gamma' \]

\[ m \sim -ic_\uparrow c_\downarrow + ic_\downarrow c_\uparrow; \quad \tilde{\sigma}_{xy} = k \text{sgn}(m) \]

Imaginary s-wave pairing mass
Interaction mediated instability on TSC surface due to multifractality

\[ H_I = U \int d^2r \left[ m(\mathbf{r}) m(\mathbf{r}) - 4 \mathbf{S}(\mathbf{r}) \cdot \mathbf{S}(\mathbf{r}) \right] + \ldots \text{(irrelevant)} \]

- **LDOS multifractal spectrum** \([\text{Sp}(2n)_k \text{ primary fields}]\)

\[ \tilde{\tau}(q) = 2(q - 1) + \Delta_q - q\Delta_1 = (q - 1) \left[ 2 - \frac{1}{2(k+1)} q \right] \]

For \( U \) interaction, \( \Delta_2^{(U)} = \Delta_2 = 0, \ \Delta_1 = 1/2(k + 1) \)

\[ \frac{dU}{dl} = (\Delta_1 - \Delta_2)U = \frac{1}{2(k + 1)} U + O(U^2) \ \text{Relevant!} \]

Order parameters break time-reversal:

- **Spin polarization** \( \mathbf{S} = c^\dagger \frac{\mathbf{S}}{2} c \)

- **Imaginary s-wave pairing mass** \( m \sim -ic^\dagger c^\dagger + ic\downarrow c\uparrow; \ \tilde{\sigma}_{xy} = k \text{sgn}(m) \)

Class C Spin QHE
Interaction mediated \textit{instability} on TSC surface due to multifractality

\[ H_I = U \int d^2 r \left[ m(r) m(r) - 4 \vec{S}(r) \cdot \vec{\tilde{S}}(r) \right] + \ldots \]

- LDOS multifractal spectrum \([\text{Sp}(2n)_{k}\text{ primary fields}]\)

\[ \tilde{\tau}(q) = 2(q - 1) + \Delta_q - q \Delta_1 = (q - 1) \left[ 2 - \frac{1}{2(k+1)} q \right] \]

For \(U\) interaction, \(\Delta_2^{(U)} = \Delta_2 = 0, \ \Delta_1 = 1/2(k + 1)\)

\[ \frac{dU}{dl} = (\Delta_1 - \Delta_2) U = \frac{1}{2(k + 1)} U + O(U^2) \]

\textbf{Physics:}
Disorder induces multifractal enhancement in energy-resolved
- spin \(\vec{S}(\varepsilon, r)\)
- pairing mass \(m(\varepsilon, r)\)
“LDOS” fluctuations
Short-range LDOS fluctuations can enhance interaction effects

\textbf{Order parameters break time-reversal:}

- \textbf{Spin polarization} \(\vec{S} = c^\dagger \frac{\vec{\sigma}}{2} c\)

- \textbf{Imaginary s-wave pairing mass} \(m \sim -ic^\dagger c^\downarrow + ic^\downarrow c^\uparrow; \ \tilde{\sigma}^{xy}_s = k \text{sgn}(m)\)

\textbf{Class C Spin QHE}
Interaction mediated \textit{instability} on TSC surface due to multifractality

\[ H_I = U \int d^2 r \left[ m^2(r) - 4 \vec{S} \cdot \vec{S}(r) \right] \]

\[ \frac{dU}{dl} = (\Delta_1 - \Delta_2)U = \frac{1}{2(k + 1)} U \]

- Due to disorder, \textit{interactions are relevant!} (multifractal enhancement)

- Expect \textit{time-reversal breaks} spontaneously.

(A) \( U \to +\infty : \langle \vec{S} \rangle \neq 0 \)

- (Local) ferromagnetic order
- Spin sym broken down to U(1)
- Unitary class A
- No gap in clean limit
- Anderson insulator with dirt (unless IQHP)

(B) \( U \to -\infty : \langle m \rangle \neq 0 \)

- Imaginary s-wave pairing of surface QPs
- Spin SU(2) symmetry \textit{preserved}
- Gap opens in clean limit
- Insulating plateau of the Class C Spin QHE

Order parameters break time-reversal:

- Spin polarization \( \vec{S} = c^\dagger \frac{\vec{\sigma}}{2} c \)

- \textit{Imaginary s-wave pairing mass} \( m \sim -ic^\dagger c_\downarrow + ic_\downarrow c^\dagger ; \quad \tilde{\sigma}_{xy}^{\text{spin}} = k \text{ sgn}(m) \)

Class C Spin QHE
Due to disorder, interactions are relevant! (multifractal enhancement)

Expect time-reversal breaks spontaneously.

Gapless Andreev states for the class CI 3D TSC (probably) do not exist.

::: Interactions plus disorder can break surface topological protection.

\[ H_I = U \int d^2 r \left[ m^2(r) - 4 \vec{S} \cdot \vec{S}(r) \right] \]

\[ \frac{dU}{dl} = (\Delta_1 - \Delta_2)U = \frac{1}{2(k+1)}U \]

Foster and Yuzbashyan 2012
**Key tools:** non-abelian bosonization, conformal embedding

**Conformal Field Theory:** $\text{SO}(4 \, n \, k)_1 \supset \text{Sp}(2 \, n)_k \oplus \text{Sp}(2 \, k)_n$

“The normal life”: Diagrammatics

1. **Self-energy rainbows (SCBA)** $\rightarrow$ Finite (non-zero, non-divergent) DOS
2. **Diffusons and Cooperons via ladders** (Gaussian fluctuations)
3. **Hikami box** (1 loop), higher order corrections
   
   [non-linear sigma model (NLsM) Wegner 1979]
4. **Add interactions** (Altshuler and Aronov 1979, Finkelstein 1984)
**Key tools:** non-abelian bosonization, conformal embedding

**Conformal Field Theory:** $\text{SO}(4 \ n \ k)_1 \supset \text{Sp}(2 \ n)_k \oplus \text{Sp}(2 \ k)_n$

“The normal life”: Diagrammatics

1. **Self-energy rainbows (SCBA) → Finite (non-zero, non-divergent) DOS**
2. **Diffusons and Cooperons via ladders (Gaussian fluctuations)**
3. **Hikami box (1 loop), higher order corrections**
   
   [non-linear sigma model (NLsM) Wegner 1979]
4. **Add interactions** (Altshuler and Aronov 1979, Finkelstein 1984)

**For 2D Dirac fermions, program fails at step 1.**

Integrating-out 2D Dirac fermions does not give controlled derivation of NLsM:

- can miss WZW/topological terms
- NLsM in strongly-coupled regime

**Check:** Many valleys, sigma model

Nersesyan, Tsvelik, Wenger 1994; Aleiner and Efetov 2006
Key tools: non-abelian bosonization, conformal embedding

Conformal Field Theory: \( SO(4 \, n \, k) \supset Sp(2 \, n) \oplus Sp(2 \, k) \)

Alternative path to the sigma model:

1. Bosonize, embed, fractionalize: \( SO(4 \, n \, k) \rightarrow Sp(2 \, n) \)

2. Bosonize: \( Sp(2 \, n) \) = principal chiral NLsM plus WZW term

3. Take large-k limit (many valleys)
   - (Zero Temp, Landauer) Conductance \( G \sim k \, e^2/h \) weak coupling!
   - Neglect WZW term (unimportant for large \( k \) “classical” limit)
   - Incorporate interactions for hydrodynamic modes

C.f. Altland, Simons, Zirnbauer 2002
Interacting (Finkel’stein) NLsM for class CI

\[ S = \frac{1}{2\lambda} \int \mathbf{r} \left[ \nabla \hat{Q}^\dagger \cdot \nabla \hat{Q} \right] - \hbar \int \mathbf{r} \left[ |\hat{\omega}_N| \hat{\Sigma}^3 (i\hat{Q} - i\hat{Q}^\dagger) \right] + k\Gamma_{WZW} \]

\[ + \Gamma_t \sum_a \int_{\tau, \mathbf{r}} \left( \text{Tr}_s \left\{ \hat{S} \left[ \hat{Q}_{aa}(\tau, \tau) - \hat{Q}_{aa}^\dagger(\tau, \tau) \right] \right\} \right)^2 \]

\[ + \Gamma_c \sum_a \int_{\tau, \mathbf{r}} \left\{ \text{Tr}_s \left[ \hat{Q}_{aa}(\tau, \tau) + \hat{Q}_{aa}^\dagger(\tau, \tau) \right] \right\}^2 \]

• “Resistance” \( \lambda = 8\pi/k \)

• Interaction couplings:

1. Repulsive spin-triplet exchange: \( \Gamma_t > 0 \)

2. Repulsive Cooper channel: \( \Gamma_c > 0 \)
1 loop RG: Dell’Anna (2006) (ignoring WZW term)

\[
\frac{d\gamma_t}{dl} = -\frac{\lambda}{2}\gamma_c (1 - \gamma_t) (1 - 2\gamma_t)
\]

\[
\frac{d\gamma_c}{dl} = \frac{\lambda}{2} \{ -3\gamma_t - 2\gamma_c + 3\gamma_c [\log(1 - \gamma_t) + \gamma_t] \} - \gamma_c^2
\]

Linearize:

\[
\frac{d\ln \gamma_r}{dl} = \frac{\lambda}{2}, \quad \frac{d\ln \gamma_i}{dl} = -\frac{3\lambda}{2}, \quad \gamma_r \equiv \gamma_c - 3\gamma_t, \quad \gamma_i \equiv \gamma_c + \gamma_t
\]

CFT: In \(k \rightarrow \infty\) limit, two interactions become marginal

\[H_I = \int d^2r \left\{ U \left[ m^2(r) - 4\vec{S} \cdot \vec{S}(r) \right] + W \mathcal{O}_W \right\}
\]

\[
\frac{d\ln U}{dl} = \frac{1}{2(k + 1)}, \quad \frac{d\ln W}{dl} = -\frac{3}{2(k + 1)}
\]

- Difference of repulsive Cooper pairing, spin exchange interactions relevant
- Using full 1-loop RG, metal is unstable to either infinite spin exchange \((\text{Stoner} \ \gamma_t \rightarrow -\infty)\) or attractive Cooper pairing \((\gamma_c \rightarrow -\infty)\)
Including feedback on resistance $\lambda$

does not change the outcome:

Dell’Anna 2006, Foster unpublished

$$\frac{d\lambda}{dl} = \lambda^2 \left[1 - (\lambda k)^2\right] + \lambda^2 \left\{ 3 \left[1 + \left(\frac{1 - \gamma_t}{\gamma_t}\right) \ln (1 - \gamma_t)\right] - \frac{\gamma_c}{2} \right\}$$
• New avenues for Anderson localization physics: local probes and LDOS fluctuations on the surface of topological insulators and superconductors

• Combined effects of disorder plus interparticle interactions can destroy topologically-protected surface states

• 2 other TSC classes in 3D with WZW-CFT surface states:
  a) Class DIII: no spin symmetry, TRI (e.g., $^3$He B, CuBi$_2$Se$_3$?)
  b) Class AIII: spin U(1) symmetry, TRI (e.g., spin-triplet p-wave)
a) Class DIII: no spin symmetry, TRI (e.g., $^3$He B, CuBi$_2$Se$_3$?)

- $k = 1$ valley ($^3$He B): **No disorder! Stable.**
- $k = 2$ valleys: Same as AllI $k = 1$ (Abelian VP). **Stable.**
- $k > 2$ valleys: $\text{SO}(n \, k)_1 \supset \text{SO}(n)_k \oplus \text{SO}(k)_n$
- WZW CFT is *already unstable* in absence of interactions
- Flow to weakly-coupled diffusive metal

\[
\frac{d\lambda}{dl} = -2\lambda^2 \left[ 1 - (k\lambda)^2 \right]
\]

- FNLsM: Pairing interactions only (spin not hydrodynamic)

- Expect usual BCS instability for attractive pairing potential (Anderson’s theorem)

**Surface pairing gap:** *imaginary pairing* (broken TRI)
b) Class AllI: spin U(1) symmetry, TRI (e.g., spin-triplet p-wave)

- $k = 1$ valley (Abelian VP): Stable for weak disorder, interactions
- $k > 1$ valleys: $\text{U}(1) \oplus \text{SU}(n,k)_1 \supset \text{U}(1) \oplus \text{SU}(n)_k \oplus \text{SU}(k)_n$
- Abelian disorder parameter $\lambda_A$ (z-spin current disorder)
- Triplet exchange, singlet pairing interactions

- Interactions irrelevant at non-zero $\lambda_A$
- Large $k$ matches FNLSM

Foster and Ludwig 2006, Dell’Anna 2006

Foster (unpublished)
b) Class AllI: spin U(1) symmetry, TRI (e.g., spin-triplet p-wave)

- $k = 1$ valley (Abelian VP): Stable for weak disorder, interactions
- $k > 1$ valleys: $\mathbf{U(1)} \oplus \mathbf{SU(n\ k)}_1 \supset \mathbf{U(1)} \oplus \mathbf{SU(n)_k} \oplus \mathbf{SU(k)_n}$
- Abelian disorder parameter $\lambda_A$ (z-spin current disorder)
- Triplet exchange, singlet pairing interactions
- **Feedback effect**: generic instability of non-interacting surface

\[
\frac{d\lambda}{dl} = \lambda^2 \left( 2 \left[ 1 + \left( \frac{1 - \gamma_t}{\gamma_t} \right) \ln (1 - \gamma_t) \right] - \gamma_c \right)
\]

\[
\frac{d\lambda_A}{dl} = \lambda^2 \left[ 1 - (k\lambda)^2 \right] + 2\lambda_A \frac{d\ln \lambda}{dl}
\]


Foster (unpublished)